DETERMINATION OF STOCHASTIC LOAD CONDITIONS ON BAR-TYPE MODELS

By

P. FUTÓ and A. KERESZTES

Department of Mechanics, Technical University Budapest

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Presented by Prof. Dr. P. MICHELBERGER

1. Introduction

In general engineering practice the procedure of dimensioning is as follows:

a) Determination of load parameters, from them determination of stresses and ultimately that of the stress condition in all points of the structure.

b) Experimental determination of the structural material characteristics with characteristics of the same units as the load parameters bearing in mind that the test results refer to uniaxial stress conditions.

c) Co-ordination of the two parameter systems.

The above characteristics are chosen in general as deterministic ones and the appropriate safety is achieved by safety factors leading to either over- or under-dimensioning.

Recently, more realistic design methods are sought for. One fundamental criterion of taking real conditions into consideration is to examine the random character of loads.

2. Stresses on Bar Systems

Structures that can be modelled with bars (e.g., the motor vehicle chassis) are generally subject to the following stresses:

$$M_v = M_{det} + M_{kin} + M_{st} + M_{din} \tag{1}$$

where:

M_v — the sum of working stresses on the structure

M_{det} — constant load of the structure (e.g., dead load)

M_{kin} — so-called kinematic stresses due to production and assembly inaccuracies

M_{st} — random stresses due to external loads

M_{din} — random stresses affecting the other stresses (e.g., unevenness of road surface)

In earlier analyses, the sum of working stresses has been determined with loads considered as deterministic. Special analyses have been concerned with the determination of these stresses by means of considering dimensional inaccuracies as possible values of a random vector variable [1].

This paper will be concerned with stresses due to external loads. Considering external load values to be constant, stresses in a hyperstatic structure with n redundancies are expressed by:

$$\mathbf{M}(s) = [\mathbf{E} - \mathbf{M}^*(\mathbf{MRM}^*)^{-1}\mathbf{MR}] \cdot \mathbf{M}_0 = \mathbf{A} \cdot \mathbf{M}_0(s) \tag{2}$$

where:

$\mathbf{M}(s)$ — column vector of stresses in different sections of the structure caused by external loads;
$\mathbf{E}$ — unit matrix;
$\mathbf{M}$ — matrix of stresses caused by unit internal forces in the structure made statically determinate;
$\mathbf{R}$ — spring matrix depending on material characteristics and geometry of the structure;
$\mathbf{M}_0(s)$ — column vector of stresses in the statically determinate structure due to external loads.

Relationship (2) is also valid for statically determinate beams in the form:

$$\mathbf{M}(s) = \mathbf{M}_0(s) \tag{3}$$

Since $\mathbf{M}_0(s)$ can be produced as a linear combination of external loads, (3) can be written as:

$$\mathbf{M}(s) = \mathbf{AB}(s)\mathbf{F} = \mathbf{C}(s)\mathbf{F} \tag{4}$$

where

$$\mathbf{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_i \\ \vdots \\ F_j \\ \vdots \\ F_k \end{bmatrix} \tag{5}$$

the column vector containing the external loads.

$\mathbf{B}(s)$ — the matrix constructed on the basis of the structure geometry converting loads into stresses;

$\mathbf{C}(s)$ — transforming matrix constructed on the basis of material characteristics and geometry of the structure.

From (4) it is obvious that the sum of working stresses can be produced as a linear combination of the external loads. Choosing the number of sections high enough, the stress-function — either linear or parabolic — can be replaced by stresses chosen as constant in each section s.

Let the possible values of the probability vector variable

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_i \\ \vdots \\ \xi_k \end{bmatrix} \tag{6}$$

be those assumed by F. Hence, possible values of ξ_i are the values of force F_i.

In knowledge of the distribution functions of the independent ξ_i the distribution function of the linear combinations of the ξ_i can be determined by convolution. In general, however, the distribution functions of the ξ_i $(i = 1, 2, \ldots, k)$ and of the — in case of independent random variables, respectively, are not available. Therefore the stress values will be attempted to be assessed from empirical values.

3. Estimating the deviation from the expected stress by means of Tchebyshev inequality

The chief advantage of the assessment is that neither the load distribution function nor the dependence degree of the ξ_i needs to be known. Force values $F_i/i = 1, 2, \ldots, k)$ at a given time will be determined on N structures subject to the same conditions.

Compiling measurements in a matrix **T**:

$$\mathbf{T} = \begin{bmatrix} t_{11} & \ldots & t_{1j} & \ldots & t_{1N} \\ \vdots & & & & \\ t_{k1} & \ldots & t_{kj} & \ldots & t_{kN} \end{bmatrix} = \begin{bmatrix} \overset{1}{t} & \overset{2}{t} & \overset{j}{t} & \overset{N}{t} \end{bmatrix}, \tag{7}$$

where:

t_{ij} — value of the i-th force acting on the j-th structure, realizations of ξ_1 being:

$$\xi_i = [t_{i1}, t_{i2}, \ldots t_{ij}, \ldots t_{iN}]. \tag{8}$$

Be the elements of $\mathbb{C}$ occurring in (4):

$$\mathbb{C} = \begin{bmatrix} c_{11} & \dots & c_{1j} & \dots & c_{1k} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & c_{ij} & \dots & c_{ik} \\ \vdots & & \vdots & & \vdots \\ c_{s1} & \dots & c_{sj} & \dots & c_{sk} \\ \vdots & & \vdots & & \vdots \\ c_{p1} & & c_{pj} & & c_{pk} \end{bmatrix} = \begin{bmatrix} c(1) \\ \vdots \\ c(i) \\ \vdots \\ c(s) \\ \vdots \\ c(p) \end{bmatrix} \tag{9}$$

where:

p — the number of structure sections.

Using these symbols, the stress in the s-th section of the structure is expressed by:

$$M(s) = \mathbb{C}(s) \cdot \mathbb{F}\,. \tag{10}$$

Be $M_j(s)$ the j-th stress momentum due to external load in the s-th section of the structure.

Using these symbols, the probability for the stress due to external load in the s-th section of the structure at time t not to deviate from the expected value by more than specified, can be determined by the *Tchebyshev* inequality:

$$P[|M(s) - M_1(s)| < \lambda_s \sqrt{M_2(s)}] \geqslant 1 - \frac{1}{\lambda_s^2}\,. \tag{11}$$

4. Producing higher empirical momentums

The *Tchebyshev* inequality contains first and second central momentums. These can be produced as follows:

$$M_1(s) = \sum_{i=1}^{k} c_i(s) \left(\frac{1}{N} \sum_{n=1}^{N} \frac{n}{t} \right) = \Phi_1 \left(\frac{1}{N} \sum_{n=1}^{N} \frac{n}{t} \right), \tag{12}$$

where:

$c_i(s)$ — the i-th element in the s-th row of $\mathbf{C}$
n/t — the n-th column of $\mathbf{T}$
Φ_1 — the operator of scalar multiplication.

The second central momentum is:

$$\begin{aligned} M_2(s) &= \sum_{i=1}^{k} \sum_{j=1}^{k} c_i(s)\, c_j(s) \left[\frac{1}{N} \sum_{n=1}^{N} \begin{pmatrix} n & & n \\ t & \mathrm{o} & t \end{pmatrix} - \frac{1}{N^2} \sum_{n=1}^{N} \sum_{l=1}^{N} \begin{pmatrix} n & & l \\ t & \mathrm{o} & t \end{pmatrix} \right] = \\ &= \Phi_2 \left[\frac{1}{N} \sum_{n=1}^{N} \begin{pmatrix} n & & n \\ t & \mathrm{o} & t \end{pmatrix} - \frac{1}{N^2} \sum_{n=1}^{N} \sum_{l=1}^{N} \begin{pmatrix} n & & l \\ t & \mathrm{o} & t \end{pmatrix} \right], \end{aligned} \tag{13}$$

where:

Φ_2 — an operator multiplying the dyad by a row vector from the left and by a column vector from the right, i.e., producing a scalar.

The *Tchebyshev* inequality yields a rather pessimistic estimation. Choosing $\pm 3\sqrt{M_2(s)}$, that is, $\lambda_s = 3$ then from (11):

$$1 - \frac{1}{3^2} = \underline{0{,}8888}.$$

Therefore an estimation method offering closer approximation has to be found. Closer estimations known from literature require the knowledge of the central stress momentums of higher order. By definition, the third central momentum is as follows:

$$M_3(s) = E\left[\sum_{i=1}^{k} c_i(s)\,\xi_i - \sum_{i=1}^{k} c_i(s)\,E(\xi_i)\right]^3 \tag{14}$$

where:

$E(\xi_i) = m_i$ means the expected value of ξ_i.
By virtue of the operational rules of polinomial multiplication and of expected values:

$$\begin{aligned} M_3(s) = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{l=1}^{k} c_l(s)\cdot c_i(s)\cdot c_j(s) E\big[(\xi_i - m_i)\cdot(\xi_j - m_j)\cdot \\ \cdot(\xi_l - m_l)\big] = \Phi_3\big[E\{(\xi - m)(\xi - m)(\xi - m)\}\big], \end{aligned} \tag{15}$$

where:

$E[(\xi_i - m_i)(\xi_j - m_j)(\xi_l - m_l)]$ is element t_{ijl} of a cubic matrix of $k\times k\times k$ size.

Taking all values of the running co-ordinates into consideration, a cubic matrix is obtained with the third central momentum of the components of random vector variable ξ along the principal body diagonal, the other elements meaning characteristics of the dependence of individual components. Φ_3 — operator transforming the cubic matrix into one numerical value.

The multiplications will be done by vector operations, providing that the double dyadic multiplication $a \circ b \circ c$ can be done in the following sequence: first, dyadic multiplication of column vector a by row vector b and dyadic multiplication of the obtained matrix by vector c perpendicular to the plane. Accordingly, the third central momentum of the stress due to external load in the s-th section of the structure at time t:

$$M_3(s) = \Phi_3\left[\frac{1}{N}\sum_{n=1}^{N}\left(\overset{n}{t}\circ\overset{n}{t}\circ\overset{n}{t}\right) - \frac{1}{N^2})\sum_{n=1}^{N}\sum_{p=1}^{N}\left(\overset{n}{t}\circ\overset{n}{t}\circ\overset{p}{t}\right) - \right.$$
$$- \frac{1}{N^2}\sum_{n=1}^{N}\sum_{p=1}^{N}\left(\overset{n}{t}\circ\overset{p}{t}\circ\overset{n}{t}\right) - \frac{1}{N^2}\sum_{n=1}^{N}\sum_{p=1}^{N}\left(\overset{n}{t}\circ\overset{n}{t}\circ\overset{n}{t}\right) + \tag{16}$$
$$\left. + \frac{2}{N^3}\sum_{n=1}^{N}\sum_{p=1}^{N}\sum_{r=1}^{N}\left(\overset{n}{t}\circ\overset{p}{t}\circ\overset{r}{t}\right)\right]$$

where:

$\overset{i}{t}$ — the i-th column vector of matrix $\mathbf{T}$.

Depending on what kind of operator $\Phi_3(s)$ is applied to transform the cubic matrix into a scalar one, the third central stress momentum of the load will be obtained in different sections of the structure. The procedure described in (16) is rather laborious, while the cubic matrix is characteristic of any section of the structure, thus yielding valuable information of the dependence method.

Be

$$\eta_i = \xi_i - m_i \tag{17}$$

where the realizations of η_i are

$$\eta_i = \{d_{i1}, d_{i2}, \ldots, d_{ij}, \ldots d_{iN}\} \quad i = 1, 2, \ldots, k \tag{18}$$

and

$$d_{ij} = t_{ij} - m_i. \tag{19}$$

Thus, the new element will be generated by substracting from each element of the matrix $\mathbf{T}$ of external loads the expected value corresponding to its row. Thereby:

$$M_3(s) = \Phi_3\left[\frac{1}{N}\sum_{n=1}^{N}\left(\overset{n}{d}\circ\overset{n}{d}\circ\overset{n}{d}\right)\right] = \frac{1}{N}\sum_{n=1}^{N}\left(\sum_{i=1}^{k} c_i(s)\, d_{in}\right)^3. \tag{20}$$

Thus, the empirical third momentum of the stress in the s-th section at time t can also be determined by multiplying the realizations of the random variables reduced to zero expected value by the transforming row vector for the s-th section. In this procedure, however, no direct information of the external load momentums and of their independence is obtained and only stress momentums for the single section s will be available. Similarly to the above, the fourth stress momentum for the s-th section is, by definition:

$$M_4(s) = \frac{1}{N}\sum_{n=1}^{N}\left(\sum_{i=1}^{k} c_i(s)\, d_{in}\right)^4. \tag{21}$$

7. Estimation of the deviation from the expected stress value by means of the method of A. WALD

A. Wald [2] made use of momentums $M_1(s)$; $M_2(s)$, $M_4(s)$ to develop the following estimation method:

$$P\left[|M(s) - M_1(s)| < d\right] \geqslant a_d \tag{22}$$

where:

$\lvert M(s) - M_1(s)\rvert$	—	the deviation of actual stresses from expected ones
d	—	arbitrarily selected number for the stress dimension
a_d	—	the lower limit of the probability for the above event.

Analysis yields two possible cases:
a) For

$$\frac{M_2(s)}{d^2} \leq \frac{M_4(s)}{d^4} \tag{23}$$

then

$$a_d = 1 - \frac{M_2(s)}{d^2} \tag{24}$$

i.e., the *Tchebyshev* inequality.
b) For

$$\frac{M_2(s)}{d^2} > \frac{M_4(s)}{d^4} \tag{25}$$

then

$$a_d = 1 - \frac{M_4(s) - M_2^2(s)}{(d^2 - M_2^2(s) + M_4(s) - M_2^2(s)} . \tag{26}$$

The following short numerical example proves the method of *A. Wald* to yield an estimation closer to the theoretical value. In the confidence interval $\pm 3\sqrt{M_2(s)}$ let us examine, while assuming normal distribution, the result by the described estimation method.
Be:

$$M_1(s) = 0$$

$$M_2(s) = 1$$

$$M_4(s) = 3$$

$$d = 3\sqrt{M_2(s)} = 3$$

$$\frac{M_2(s)}{d^2} > \frac{M_4(s)}{d^4}$$

$$\frac{1}{3^2} > \frac{3}{3^4} \text{ is valid and so}$$

$$a_d = 1 - \frac{M_4(s) - M_2^2(s)}{d^2 - M_2^2(s) + M_4(s) - M_2^2(s)} =$$

$$= 1 - \frac{3-1}{(3^2-1)^2 - 3 - 1} = 0{,}9697.$$

Therefore
$P[|M(s) - M_1(s)| < d] \geqslant 0.9697$, greater than 0,888 obtained by means of the *Tchebyshev* inequality and closely approximating the theoretical value 0,997.

8. Estimation of the deviation from the expected stress value by means of the multidimensional Tchebyshev inequality

The above analysis determined the probability of the structure not to fail in its different s-th sections. Nevertheless, safety against failure of the structure in its different sections is not equivalent to the safety of the structure as a whole. This were true only by meeting the overall distribution function of the stresses in each section, or, in case of estimation, of the overall criterion. The whole structure — considering each section to be linear — can be assessed by means of the multidimensional *Tchebyshev* inequality [3]:

$$P\left(\left|\frac{M(1)-M_1(1)}{\sqrt{M_2(1)}}\right| < \lambda_1, \ldots \left|\frac{M(s)-M_1(s)}{\sqrt{M_2(s)}}\right| < \lambda_s, \ldots \right.$$
$$\left. \ldots \left|\frac{M(p)-M_1(p)}{M_2(p)}\right| < \lambda_p\right) \geqslant 1 - \left[\frac{\sqrt{u} + \sqrt{(p-1)\left(\sum_{s=1}^{p}\lambda_s^{-2} - u\right)}}{p}\right]^2 \tag{27}$$

λ_s — the value of permissible deviation

$$u = \sum_{s=1}^{p} \lambda_s^{-2} + 2\sum_{i<j}\sum \varrho_{ij}\,\lambda_i^{-1} \cdot \lambda_j^{-1} \tag{28}$$

$$\varrho_{ij} = \text{corr}\ [M(i),\ M(j)]\,. \tag{29}$$

Introducing the notation:

$$\lambda_s \cdot \sqrt{M_2(s)} = Z_s \tag{30}$$

further

$$\mathbf{D} = [\text{cov}(i, j)] = E\big[\big(M(i) - M_1(i)\big) \cdot \big(M(j) - M_1(j)\big)\big] \tag{31}$$
$$i = 1,\ 2,\ \ldots;\ p$$
$$j = 1,\ 2,\ \ldots,\ p$$

the matrix of the standard deviation, then

$$u=\left[\frac{1}{z_1},\ \ldots,\ \frac{1}{z_p}\right]\left[\begin{array}{c}\mathbf{D}\\ \text{matrix of the}\\ \text{standard}\\ \text{deviation}\end{array}\right]\left[\begin{array}{c}\frac{1}{z_1}\\ \cdot\\ \cdot\\ \cdot\\ \frac{1}{z_p}\end{array}\right]. \tag{32}$$

Hence:

$$P(|M(1)-M_1(1)|<Z_1,\ \ldots\ |M(p)-M_1(p)|<Z_p)\geqslant 1- \tag{33}$$
$$-\left[\frac{\sqrt{u}+\sqrt{(p-1)\sum\limits_{s=1}^{p}\frac{M_2(s)}{Z_s^2}-u}}{p}\right]^2.$$

If the expected value is zero and a two-dimensional case is investigated, then Equation (33) simplifies into:

$$P\left(|M(1)|<Z_1,\ |M(2)|<Z_2\right)>1-\frac{1}{2}\left[\frac{M_2(1)}{Z_1^2}+\frac{M_2(2)}{Z_2^2}+\right. \tag{34}$$
$$\left.+\sqrt{\left[\frac{M_2(1)}{Z_1^2}+\frac{M_2(2)}{Z_2^2}\right]^2-\frac{4\left[\operatorname{cov}M(1),M(2)^2\right.}{Z_1^2\cdot Z_2^2}}\right].$$

Since, in case of independence, the covariance is zero, the term in the relationship to be substracted increases, thus leading to a more pessimistic estimation. That is, in case of dependence the probability of the structure to hold will increase.

Conclusions

1. The n-th empirical central momentum of stresses due to external loads can be produced by means of an n-dimensioned matrix.

2. Higher-order momentums may yield estimations closer than the Tchebyshev inequality, for the same confidence intervals.

3. Realistic estimation of the probability of the safety against failure of a whole structure is offered only by multidimensional inequalities.

Summary

Structures are generally exposed to stochastic loads. Several methods are presented of siresssang the stresses if the distribution function of these loads is unknown.

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* In Hungarian.

Péter FUTÓ
Albert KERESZTES } 1450 Budapest, P.O.B. 93 Hungary