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ATTITUDE DYNAMICS OF A SATELLITE ON A CIRCULAR AND ELLIPTIC LOW EARTH ORBIT

Péter B. BÉDA

Department of Mechanics
Faculty of Transportation Engineering
Technical University of Budapest

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Abstract

The paper deals with the effect of the length on the type and stability of the inplane attitude motion of a dumbbell satellite moving on circular and elliptic equatorial Low Earth Orbits (LEO) by which the air drag also has a weak influence. For a circular LEO, a saddle-node bifurcation is found at some critical value of the length. This investigation can be performed analytically using imperfect bifurcation theory. When the orbit is elliptic using the results from the circular case, numerical simulation is used to approach the phase trajectories for sub- and supercritical values of the length. Depending on the order of the orbit eccentricity, three kinds of behaviour seem to be possible.

Keywords: dumbbell, satellite, imperfect bifurcation, chaos.

1. Introduction

The usage of space shuttles is a challenging possibility for the application of large space structures. One of the first of them is the Tethered Satellite System (TSS) consisting of two subsatellites connected by a very long thin rope or cable called the tether [15]. Its length could even be up to 200 km. The dynamics of such a mechanical system is an exciting new field for researchers and engineers [14, 16].

The motion of a satellite with finite size can be divided into an orbiting motion of the centre of mass and the rotations about it commonly referred to as librations. Librations can occur in and out of the plane of the orbit called inplane and out-of-plane motions, respectively. The field dealing with librational dynamics and stability of Earth orbiting systems is called attitude dynamics.

The simplest model of a TSS is the well-known dumbbell satellite. Instead of the elastic cable it assumes a massless rigid connection of the subsatellites. In some cases of the literature [1, 2, 5] the usage of that simplification can be justified.

This paper deals with the non-linear and possibly chaotic oscillations of the satellite system. In the literature of the deterministic chaos, one of the most widely mentioned examples is the excitation of a conservative system having a double well potential. It is shown by using the theory of Smale's horseshoe that a chaotic motion appears at the saddle-type equilibrium point of such systems [17].

When the effect of the air drag can be omitted and the dumbbell satellite has a perfect circular equatorial orbit, its inplane attitude dynamics has a double well potential, too. The question under consideration in this paper is, what happens if both air drag and the excitation of the orbit eccentricity are present. Will the chaotic behaviour of the undamped case survive? Another question arisen from the previous one is, how the excited system behaves around this critical length.

There are two possible principal ways to give the answers. One of them is an analytical approach, that is, to do all the possible simplifications in the equation of motion to find an analytical solution to some equation possessing the same qualitative behaviour as the original one. It is performed in the case of a circular orbit. The mathematical tool of it is called the bifurcation theory [3, 11, 19].

Unfortunately, the same analytical investigation would be very complicated for the eccentric case being the main subject of our interest. To an eccentric orbit, the second possible approach, the numerical simulation will be applied, and the results of the circular orbiting motion will show for which values of the parameters it would be interesting to do the simulation.

2. The Equation of Motion

The position of the centre of mass of a satellite can be given by vector $\mathbf{R}_c$ according to the centre of the Earth.

By using the so-called true anomaly f being the angle between the recent position of the centre of mass $\mathbf{R}_c$ and the position vector $\mathbf{P}$ of the perigee for a Keplerian orbit, the derivative of it is [13]

$$\dot{f} = \frac{\sqrt{GM_e}}{(a(1-e^2))^{\frac{3}{2}}}(1 + e\cos f)^2 \,, \tag{1}$$

where a is the semimajor axis of the ellipse of the orbit, e is the eccentricity, M_e is the mass of the Earth, and G is the gravitational coefficient. The radial and tangential components of the orbital velocity are

$$v_r = \frac{\sqrt{GM_e}}{\sqrt{a\,(1-e^2)}} e\sin f\,, \qquad v_t = \frac{\sqrt{GM_e}}{\sqrt{a\,(1-e^2)}}\,(1 + e\cos f)\;. \tag{2}$$

To determine the orientation of the orbit, three more constants are also required [18]: the longitude ascending node Ω, the inclination I and the argument of the perigee ω.

In obtaining the equations of motion of the satellite system, two types of coordinate systems will be introduced. The first one is a fixed global system X_0, Y_0, Z_0 considered to be an inertial one [4]. The second one is a local frame x, y, z moving together with the orbiting centre of mass of the satellite system. To transform a vector from the frame X_0, Y_0, Z_0 into x, y, z, rotation around Z_0 with angle $-I$, then rotation around y' with angle $-(\omega + f)$ is necessary. The variation of the inplane position of the dumbbell satellite, that is, the attitude dynamics will be described in the local frame by the angle θ.

The equation of motion of the dumbbell satellite will be obtained in the form of a Lagrange equation using the pitch angle θ as a generalized coordinate.

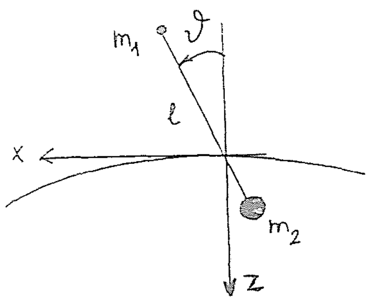

Fig. 1.

In *Fig. 1*, the subsatellites denoted by m_1, m_2 are considered as point masses, the distance of masses m_1, m_2 is l. By omitting the mass of the tether and the effect of the air drag on the orbit of the centre of mass C of the system, it moves on an elliptic orbit described by the position vector $\mathbf{R}_c$ in the inertial geocentric frame X_0, Y_0, Z_0 with velocity $\dot{\mathbf{R}}_c$. By using the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ of masses m_1, m_2 in the local coordinate system x, y, z.

$$x_1 m_1 + x_2 m_2 = 0\,, \qquad z_1 m_1 + z_2 m_2 = 0\,. \tag{3}$$

By introducing the generalized coordinate θ

$$x_1 = l \sin\theta + x_2\,, \qquad z_1 = l\cos\theta + z_2\,. \tag{4}$$

From equations (3) and (4) with notation

$$\mu = \frac{m_1}{m_1 + m_2},$$

the coordinates of the point masses will be:

$$x_1 = (1-\mu)l\sin\theta\,, \qquad z_1 = (1-\mu)l\cos\theta\,,$$
$$x_2 = -\mu l\sin\theta\,, \qquad z_2 = -\mu l\cos\theta\,. \tag{5}$$

By introducing $m = m_1 + m_2$ the kinetic energy of the dumbbell satellite will be:

$$T = \frac{1}{2}m\dot{R}_c^2 + \frac{1}{2}\left(m_1\dot{r}_1^2 + m_2\dot{r}_2^2\right). \tag{6}$$

For obtaining the kinetic energy as a function of the generalized coordinate, one should express the velocities of the point masses. If $-\dot{f}$ denotes the angular velocity of the local frame according to X_0, Y_0, Z_0,

$$\dot{\mathbf{r}}_{\mathrm{i}} = \begin{bmatrix} \dot{x}_i - \dot{f}z_i \\ 0 \\ \dot{z}_i + \dot{f}x_i \end{bmatrix}, \qquad i = 1, 2\,. \tag{7}$$

From (2)

$$\dot{R}_c^2 = \frac{GM_e}{a\,(1-e^2)}\left(1 + e^2 + 2e\cos f\right).$$

By substituting it into (6), after some simplifications:

$$T = \frac{1}{2}m\left(\frac{GM_e}{a\,(1-e^2)}\left(1 + e^2 + 2e\cos f\right) - \eta_1\eta_2 l^2\left(\dot{f} - \dot{\theta}\right)^2\right),$$

where $\eta_1 = 1-\mu$, $\eta_2 = -\mu$. Then the first part of the equation of motion has the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -\eta_1\eta_2 l^2 m\left(\ddot{\theta} + 2e\frac{GM_e}{a^3\,(1-e^2)^3}\,(1 + e\cos f)^3\sin f\right).$$

The gravitational potential of the system is

$$V = -GM_e\left(\frac{m_1}{|\mathbf{R}_c + \mathbf{r}_1|} + \frac{m_2}{|\mathbf{R}_c + \mathbf{r}_2|}\right).$$

To obtain the gravitational forces, one should express $\mathbf{R_c}$ in the coordinate system x, y, z

$$\mathbf{R_c} = \frac{a\left(1-e^2\right)}{1+e\cos f}\begin{bmatrix}0\\0\\-1\end{bmatrix}. \tag{8}$$

By using (5) and (8)

$$\mathbf{R_c} + \mathbf{r_i} = \begin{bmatrix}\eta_i l s_\theta\\0\\-\frac{a\left(1-e^2\right)}{1+ec_f}+\eta_i l c_\theta\end{bmatrix}, \qquad i = 1,\, 2\,, \tag{9}$$

where $c = \cos$, $s = \sin$. The absolute values of the radii are

$$|\mathbf{R_c} + \mathbf{r_i}| = \left(\eta_i^2 l^2 + \frac{a^2\left(1-e^2\right)^2}{(1+ec_f)^2} - 2\eta_i l c_\theta \frac{a\left(1-e^2\right)}{1+ec_f}\right)^{\frac{1}{2}}, \qquad i = 1,\, 2. \tag{10}$$

By using (10), the effect of the gravitational potential V can be taken into the Lagrange equation as

$$\frac{\partial V}{\partial \theta} = GM_e m\eta_1\eta_2 l s_\theta \frac{a\left(1-e^2\right)}{1+ec_f}\left(\left(\eta_2^2 l^2 + \frac{a^2\left(1-e^2\right)^2}{\left(1+ec_f\right)^2} - 2\eta_2 l c_\theta \frac{a\left(1-e^2\right)}{1+ec_f}\right)^{-\frac{3}{2}} - \left(\eta_1^2 l^2 + \frac{a^2\left(1-e^2\right)^2}{\left(1+ec_f\right)^2} - 2\eta_1 l c_\theta \frac{a\left(1-e^2\right)}{1+ec_f}\right)^{-\frac{3}{2}}\right).$$

The effect of the air drag is usually taken into consideration by aerodynamical force $\mathbf{F}$

$$\mathbf{F} = -\frac{1}{2}C_d\rho\Delta A\mathbf{v}|\mathbf{v}|,$$

where

- C_d is the coefficient of air resistance,
- ΔA is the projected area of the satellites on the flow velocity,
- ρ is the density of the atmosphere,
- $\mathbf{v}_i$ are the velocities of the satellites relative to the rotating atmosphere of the Earth.

ρ decreases exponentially with the increasing height h [7, 12], $\rho = \rho_0 e^{-\frac{h}{h_0}}$, where ρ_0, h_0 are constants for a given portion of the atmosphere.

When denoting the radius of the Earth by R_e, the atmospheric forces at the subsatellites are

$$\mathbf{F}_i = -\frac{1}{2} C_{di} \rho_0 e^{-\frac{R_c - R_e - z_i}{h_0}} \Delta A_i \mathbf{v}_i |\mathbf{v}_i| , \qquad i = 1, 2 . \tag{11}$$

Velocities $\mathbf{v}_i$ are the differences of the velocities of the subsatellites $\dot{\mathbf{R}}_c + \dot{\mathbf{r}}_i$, $(i = 1, 2)$ and the velocity of the rotating atmosphere,

$$\mathbf{v}_i = \dot{\mathbf{R}}_c + \dot{\mathbf{r}}_i - \mathbf{v}_{atmi} , \qquad i = 1, 2 .$$

From (2)

$$\dot{\mathbf{R}}_c = \frac{\sqrt{GM_e}}{\sqrt{a\left(1-e^2\right)}} \begin{bmatrix} 1 + e\cos f \\ 0 \\ -e \sin f \end{bmatrix} ,$$

then

$$\dot{\mathbf{R}}_c + \dot{\mathbf{r}}_i = \begin{bmatrix} \frac{\sqrt{GM_e}}{\sqrt{a(1-e^2)}} \left(1 + e c_f\right) - \eta_i l c_\theta \left(\dot{f} - \dot{\theta}\right) \\ -e \frac{\sqrt{GM_e}}{\sqrt{a(1-e^2)}} s_f + \eta_i l s_\theta \left(\dot{f} - \dot{\theta}\right) \end{bmatrix} , \qquad i = 1, 2 .$$

On the other hand,

$$\mathbf{v}_{atmi} = \sigma_{xyz} \times \left(\mathbf{R}_c + \mathbf{r}_i\right) ,$$

where σ_{xyz} is the angular velocity of the rotating Earth in the coordinate system x, y, z,

$$\sigma_{xyz} = \sigma \begin{bmatrix} s_I c_{\omega+f} \\ -c_I \\ -s_I s_{\omega+f} \end{bmatrix} .$$

Having done the necessary substitutions

$$\mathbf{v}_i = \sqrt{\frac{GM_e}{a\left(1-e^2\right)}} \begin{bmatrix} 1 + e c_f \\ 0 \\ -e s_f \end{bmatrix} + \frac{\sigma a \left(1 - e^2\right)}{1 + e c_f} \begin{bmatrix} -c_I \\ -s_I c_{\omega+f} \\ 0 \end{bmatrix} +$$

$$+ \eta_i l \begin{bmatrix} -c_\theta \left(\dot{f} - \dot{\theta}\right) \\ 0 \\ s_\theta \left(\dot{f} - \dot{\theta}\right) \end{bmatrix} + \eta_i l \sigma \begin{bmatrix} c_I c_\theta \\ s_I \left(s_{f+\omega} s_\theta + c_{f+\omega} c_\theta\right) \\ -c_I s_\theta \end{bmatrix} . \tag{12}$$

To introduce the aerodynamical forces into the Lagrange equation, the generalized force Q_θ is necessary

$$Q_\theta = \sum_{i=1}^{2} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \theta} .$$

By using (5) and (12)

$$Q_\theta = -\frac{1}{2}\rho_0 l e^{-\frac{R_c-R_e}{h_0}} \sum_{i=1}^{2} e^{\frac{z_i}{h_0}} C_{di}\Delta A_i |\mathrm{v}_i| \eta_i \left(\sqrt{\frac{GM_e}{a\left(1-e^2\right)}} \left(c_\theta + e c_{f-\theta}\right) - \right.$$

$$\left. - \frac{a\left(1-e^2\right)}{1+ec_f} \sigma c_\theta c_I + \eta_i l \left(\dot{\theta} - \dot{f} + \sigma c_I\right)\right) .$$

Thus the equation of the single degree-of-freedom inplane motion in an inertial system using generalized coordinate θ will be:

$$-\eta_1\eta_2 l^2 m \left(\ddot{\theta} + 2e\frac{GM_e}{a^3\left(1-e^2\right)^3}\left(1+ec_f\right)^3 s_f\right) +$$

$$+GM_e m\eta_1\eta_2 l s_\theta \frac{a\left(1-e^2\right)}{1+ec_f}\left(\left(\eta_2^2 l^2 + \frac{a^2\left(1-e^2\right)^2}{\left(1+ec_f\right)^2} - \right.\right.$$

$$\left.\left. - 2\eta_2 l c_\theta \frac{a\left(1-e^2\right)}{1+ec_f}\right)^{-\frac{3}{2}} - \left(\eta_1^2 l^2 + \frac{a^2\left(1-e^2\right)^2}{\left(1+ec_f\right)^2} - 2\eta_1 l c_\theta \frac{a\left(1-e^2\right)}{1+ec_f}\right)^{-\frac{3}{2}}\right) =$$

$$= -\frac{1}{2}\rho_0 l e^{-\frac{R_c-R_e}{h_0}} \sum_{i=1}^{2} e^{\frac{z_i}{h_0}} C_{di}\Delta A_i |\mathrm{v}_i| \eta_i \left(\sqrt{\frac{GM_e}{a\left(1-e^2\right)}} \left(c_\theta + e c_{f-\theta}\right) - \right.$$

$$\left. - \frac{a\left(1-e^2\right)}{1+ec_f} \sigma c_\theta c_I + \eta_i l \left(\dot{\theta} - \frac{\sqrt{GM_e}}{\left(a\left(1-e^2\right)\right)^{\frac{3}{2}}}\left(1+ec_f\right)^2 + \sigma c_I\right)\right) , \qquad (13)$$

where function f is the solution of the differential *Eq.* (1). The inplane motion of the dumbbell satellite can be described as a function of time by *Eqs.* (1) and (13). The first one concerns the orbiting motion of the centre of mass and can be solved independently. When having obtained function f, *Eq.* (13) characterizes the inplane attitude dynamics of the system.

From (11) and (12), it can be seen that the aerodynamical forces remain in the plane of the orbit, if inclination I is zero. Then the satellite is in a so-called equatorial orbit. By assuming $I=0$, the inplane attitude dynamics will be considered as the first simplification of the problem.

For such an equatorial orbit, the equation of motion of the dumbbell satellite is a rather complicated second order non-linear one, moreover also the solution of (1) is needed. There is no hope to find an analvtical solution.

By applying a simplification used generally in literature [5], we introduce the ratio $\varepsilon = \frac{l}{a}$ of the length of the dumbbell satellite and the semimajor axis of the orbit as a kind of dimensionless length. The maximal tether length is about 200 km, while the semimajor axis should be greater than the radius of the Earth, that is, $\varepsilon \ll 1$. By expanding into a power series in ε, the higher order terms can be neglected. As a simplified form of (13), its truncated second order Taylor expansion

$$\varepsilon\left(\ddot{f}-\ddot{\theta}\right)-\dot{f}^2\frac{3s_\theta}{1+ec_f}\varepsilon\left(c_\theta-\frac{2\mu-1}{2}\frac{1+ec_f}{1-e^2}\left(1-5c_\theta^2\right)\varepsilon\right)-$$
$$-\frac{\alpha}{2}\left(A\left(\frac{1-e^2}{1+ec_f}\right)^2\left(\left(\dot{f}-\sigma\right)^2+\dot{f}^2\frac{e^2s_f^2}{\left(1+ec_f\right)^2}\right)^{\frac{1}{2}}\cdot\right.$$
$$\cdot\left(\left(\dot{f}-\sigma\right)c_\theta+e\dot{f}s_fs_\theta\right)+B\varepsilon\frac{1-e^2}{1+ec_f}\left(\left(\left(\dot{f}-\sigma\right)^2+\dot{f}^2\frac{e^2s_f^2}{\left(1+ec_f\right)^2}\right)^{\frac{1}{2}}\cdot\right.$$
$$\cdot\left(\left(\dot{\theta}-\dot{f}+\sigma\right)+\beta c_\theta\frac{1-e^2}{1+ec_f}\left(\left(\dot{f}-\sigma\right)c_\theta+e\dot{f}s_fs_\theta\right)\right)+$$
$$+\left(\dot{\theta}-\dot{f}+\sigma\right)\left(\left(\dot{f}-\sigma\right)c_\theta+\dot{f}s_\theta\frac{es_f}{1+ec_f}\right)\left(\left(\dot{f}-\sigma\right)c_\theta+e\dot{f}s_fs_\theta\right)\cdot$$
$$\left.\left.\cdot\left(\frac{1-e^2}{1+ec_f}\right)^2\left(\left(\dot{f}-\sigma\right)^2+\dot{f}^2\frac{e^2s_f^2}{\left(1+ec_f\right)^2}\right)^{-\frac{1}{2}}\right)\right)=0 \qquad (14)$$

is obtained as equation of motion, where

$$A=\frac{C_{d1}(1-\mu)-\kappa\mu C_{d2}}{\mu(1-\mu)},\qquad B=\frac{C_{d1}(1-\mu)^2+\kappa\mu^2C_{d2}}{\mu(1-\mu)},$$

$$\alpha=\frac{\rho_0e^{-\frac{R_c-R_e}{h_0}}\Delta A_1R_c}{m},\qquad \beta=\frac{R_c}{z_0},\qquad \kappa=\frac{\Delta A_2}{\Delta A_1}.$$

There are two possibilities at this point. Firstly, an analytic calculation for the circular orbit can be performed based on the imperfect bifurcation theory. By using some analytical methods of the qualitative theory of differential equations, there could be a possibility to find results on the stability of the motions [19, 21].

Secondly, the eccentric case can be investigated by using numerical simulation and solve the equations of motion approximately by a computer. These result could be problematic [10], but probably they can have an influence on the behaviour of the motion. In this paper, both possibilities are treated.

3. The Circular Orbit

In case of a circular orbit, $e=0$, $a=R_c$, $\dot{f}=\Omega=$ constant and the equation of motion from (14) is

$$\varepsilon\ddot{\theta}+3\Omega^2\sin\theta\left(\varepsilon\cos\theta-\frac{1}{2}(2\mu-1)\left(1-5\cos^2\theta\right)\varepsilon^2\right)+$$
$$+\frac{\alpha}{2}(\Omega-\sigma)\left((\Omega-\sigma)\cos\theta A+B\varepsilon\left(\left(\dot{\theta}-\Omega\right)\left(1+\cos^2\theta\right)+\beta\cos^2\theta(\Omega-\sigma)+\right.\right.$$
$$\left.\left.+\sigma\left(\cos^2\theta+1\right)\right)\right)=0\,. \tag{15}$$

As a starting point, the possible states of equilibria are needed. By substituting $\ddot{\theta}=\dot{\theta}=0$ for the equilibrium solutions of (15)

$$3\varepsilon\Omega^2\sin\theta\left(\cos\theta-\frac{1}{2}(2\mu-1)\left(1-5\cos^2\theta\right)\varepsilon\right)+$$
$$+\frac{\alpha}{2}(\Omega-\sigma)\left((\Omega-\sigma)\cos\theta A+B\varepsilon\left(-\Omega\left(1+\cos^2\theta\right)+\beta\cos^2\theta(\Omega-\sigma)+\right.\right.$$
$$\left.\left.+\sigma\left(\cos^2\theta+1\right)\right)\right)=0\,. \tag{16}$$

First, one should truncate (16) up to the linear terms in ε

$$3\varepsilon\Omega^2\sin\theta\cos\theta+\frac{\alpha}{2}(\Omega-\sigma)^2\cos\theta A=0\,. \tag{17}$$

Eq. (17) has two kinds of physically important solutions

$$\cos\theta=0\ :\qquad \theta_1=\frac{\pi}{2}\,,\quad \theta_2=-\frac{\pi}{2}\,,$$

and

$$\theta_{3,4}=\arcsin\left(-\frac{\alpha}{6\Omega^2\varepsilon}(\Omega-\sigma)^2A\right)\,.$$

In the second case

$$D=\left|\frac{\alpha}{6\Omega^2\varepsilon}(\Omega-\sigma)^2A\right|$$

determines the possible number of the solutions. For $D>1$ 0, for $D=1$ 1, and for $D<1$ 2 solutions exist. The number of the elements in the set of solutions changes at $D=1$, that is, there exists a critical dimensionless length

$$\varepsilon_c=\frac{\alpha}{6\Omega^2}(\Omega-\sigma)^2A\,.$$

By using numerical values for the parameters from literature [20] $m_1 = 200\,\mathrm{kg}$, $m_2 = 1000\,\mathrm{kg}$, $R_c = 6578\,\mathrm{km}$, $C_{d1} = C_{d2} = 1$, $\Omega = 1.18 \cdot 10^{-3}\,\mathrm{s}^{-1}$, $\mu = 0.167$, $\beta = 32.89$, $\sigma = 7 \cdot 10^{-5}\,\mathrm{s}^{-1}$, $\alpha = 1.24 \cdot 10^{-2}$. Let $\kappa = 3$, then $A = 2.39$, $B = 5.60$.

The critical value is $\varepsilon_c = 4.65 \cdot 10^{-3}$, that means a $l_c = 30.7\,\mathrm{km}$ critical length at which the three equilibria are at the same position

$$\theta_3 = \theta_1 = \arcsin(-1) = -\frac{\pi}{2} = \theta_2 .$$

The result of the first approach can be summarized in the existence of a critical length at which the set of equilibria undergoes a static bifurcation in the sense of [6]. The following question is what change in the stability properties of the various solutions occurs at the critical length of the dumb-bell satellite. This investigation requires a truncated equation of motion.

$$\ddot{\theta} + 3\Omega^2 \sin\theta \cos\theta + \frac{\alpha}{2\varepsilon}(\Omega - \sigma)^2 \cos\theta A = 0 . \tag{18}$$

By introducing $\theta_1 = \theta$, $\theta_2 = \dot{\theta}$ as new variables into (18), a system of two first order differential equations

$$\dot{\theta}_1 = \theta_2 ,$$

$$\dot{\theta}_2 = -3\Omega^2 \sin\theta_1 \cos\theta_1 - A\frac{\alpha}{2\varepsilon}(\Omega - \sigma)^2 \cos\theta_1$$

is obtained. By linearizing them at $-\frac{\pi}{2}$, $\frac{\pi}{2}$ the eigenvalues of the matrices of coefficients are

$$\Lambda_{\frac{\pi}{2}} = \pm\sqrt{3\Omega^2 + A\frac{\alpha}{2\varepsilon}(\Omega - \sigma)^2}$$

and

$$\Lambda_{-\frac{\pi}{2}} = \pm\sqrt{3\Omega^2 - A\frac{\alpha}{2\varepsilon}(\Omega - \sigma)^2} , \qquad \varepsilon \geq \varepsilon_c ,$$

$$\Lambda_{-\frac{\pi}{2}} = \pm i\sqrt{-3\Omega^2 + A\frac{\alpha}{2\varepsilon}(\Omega - \sigma)^2} , \qquad \varepsilon < \varepsilon_c .$$

Thus in case $\varepsilon > \varepsilon_c$, both of the equilibria $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are unstable saddle-points. If $\varepsilon = \varepsilon_c$ or less, then the equilibrium $-\frac{\pi}{2}$ becomes a centre. Unfortunately, a centre is structurally unstable, that is, *Eq.* (18) is not suitable for a stability investigation [3]. To make it correct, all the terms of (15) are necessary. The additional terms will cause technical difficulties and one should apply imperfect bifurcation theory to overcome them.

Now the static bifurcation theory (in the sense of [6]) is applied to the equilibrium solutions of *Eq.* (15). The problematic part of that treatment is that the exact equilibria are impossible to find in an analytical form because of the complexity of (15). However, an approximate solution is known and localizing at this approximate solution, the method for imperfect bifurcations [9, 11] can be applied.

The only damping effect done by air drag is in the second order terms of the ε power series expansion, and it is necessary to get strict stability properties. We have also seen the necessity of these terms in the investigation of the way of the change in the number of equilibria, because structural instability means that even a small change in the equation (like a weak damping effect of the atmosphere) can cause radical changes of the bifurcation diagram. So all terms of equilibrium equation (16) are necessary, but then the equation cannot be solved analytically. The only thing we know from the previous part is that some bifurcation should happen in the vicinity of ε_c to some equilibrium existing in the vicinity of $\theta = -\frac{\pi}{2}$.

The basis of the bifurcation methods is a localization of the equations having a distinguished parameter at a solution being singular at a critical parameter value. First, one should find an appropriate parameter. Let us define $\chi = \frac{\alpha}{\varepsilon}$. Its critical value is $\chi_c = \frac{\alpha}{\varepsilon_c}$. When substituting the numerical values, $\chi_c = 2.8$. By introducing χ and $\theta = -\frac{\pi}{2} + q_1$, the power series expansion of (16) is

$$c_{01}\varepsilon\chi + c_{02}\varepsilon + c_{11}\chi q_1 + c_{12}q_1 + c_{22}\varepsilon q_1^2 + c_{31}\chi q_1^3 + c_{32}q_1^3 + \cdots = 0\,, \qquad (19)$$

where c_{ij} are constants which can be calculated by using the data of the TSS experiment. In our case, the numerical values are $c_{01} = -3.45 \cdot 10^{-6}$, $c_{02} = 1.39 \cdot 10^{-6}$, $c_{11} = 1.47 \cdot 10^{-6}$, $c_{12} = -4.18 \cdot 10^{-6}$, $c_{22} = -7.65 \cdot 10^{-6}$, $c_{31} = -0.25 \cdot 10^{-6}$, $c_{32} = 2.78 \cdot 10^{-6}$. By using the bifurcation parameter

$$\lambda = \frac{\varepsilon - \varepsilon_c}{\varepsilon_c}\,,$$

ε and χ can be expressed as functions of it

$$\varepsilon = \varepsilon_c\lambda + \varepsilon_c = (\lambda + 1)\varepsilon_c\,,$$

$$\chi = \frac{\alpha}{\varepsilon} = \frac{\alpha}{\varepsilon_c}\frac{1}{1+\lambda} = \frac{\alpha}{\varepsilon_c}\left(1 - \lambda + \lambda^2 - \dots\right).$$

Now one can use the bifurcation parameter λ and ε_c in (19). Considering only the terms up to first order in them and truncating (19) at the third order terms in q_1, an imperfect pitchfork bifurcation [8]

$$\lambda c_{12}q_1 + \left(c_{32} - \frac{c_{12}c_{31}}{c_{11}}\right)q_1^3 + \varepsilon_c\left(c_{22}q_1^2 + \left(c_{02} - \frac{c_{12}c_{01}}{c_{11}}\right)\right) = 0 \qquad (20)$$

can be recognized. Substituting numerical values after a multiplication by 10^{-6}, we obtain

$$0.50q_1^3 - \lambda q_1 - \left(9.35 + 8.51q_1^2\right) \cdot 10^{-3} = 0\,. \tag{21}$$

Fig. 2 shows the bifurcation diagram (21) in coordinates q_1, λ.

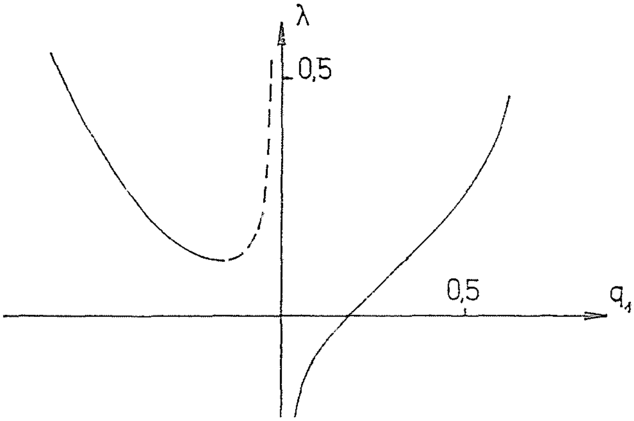

Fig. 2.

Solid line means a stable, dashed line an unstable equilibrium. These stability properties are dealt with in the next part of the paper.

For stability investigations, the use of the whole equation of motion is necessary, the equilibrium equation, that is, the $\ddot{\theta} = \dot{\theta} = 0$ simplification is unsuitable. This fact complicates the treatment, for example duplicates the number of the differential equations, because the second order equation of motion should be transformed into a system of first order equations [19]. By introducing new variables $q_1 = -\frac{\pi}{2} + \theta$, $q_2 = \dot{\theta}$ and having done the same kind of localization as in the previous part,

$$\dot{q}_1 = q_2\,,$$

$$\dot{q}_2 = -\varepsilon\left(c_{01}\chi + c_{02}\right) - q_1\left(c_{11}\chi + c_{12}\right) + q_1^2 c_{22}\varepsilon - q_1^3\left(c_{31}\chi + c_{32}\right) + \varepsilon\chi d q_2$$

is obtained, where $d = 3.85 \cdot 10^{-5}$. By using similarly parameters λ, ε_c, the equations become

$$\dot{q}_1 = q_2\,,$$

$$\dot{q}_2 = -c_{12}\lambda q_1 + \frac{c_{12}d}{c_{11}}\varepsilon_c q_2 - \left(c_{32} - \frac{c_{12}c_{31}}{c_{11}}\right) q_1^3 - \varepsilon_c \left(c_{22}q_1^2 + \left(c_{02} - \frac{c_{12}c_{01}}{c_{11}}\right)\right), \tag{22}$$

or with numerical data

$$\dot{q}_1 = q_2 \,,$$
$$\dot{q}_2 = 4.18\lambda q_1 - 0.51 q_2 - 2.07 q_1^3 + \left(3.92 + 3.56 q_1^2\right) \cdot 10^{-2} \,.$$

For investigating the stability of the bifurcated solutions, one should calculate the derivatives $\frac{\partial}{\partial q_1}$, $\frac{\partial}{\partial q_2}$ of the left hand side of (22). By writing them into matrix $\mathbb{D}$

$$\mathbb{D} = \begin{pmatrix} 0 & 1 \\ -c_{12}\lambda - 3\left(c_{32} - \frac{c_{12}c_{31}}{c_{11}}\right) q_1^2 - 2\varepsilon_c c_{22} q_1 & \frac{c_{12}d}{c_{11}}\varepsilon_c \end{pmatrix}$$

its eigenvalues $\Lambda_{1,2}$ show the Lyapunov stability of the solutions. Generally, these are

$$\Lambda_{1,2} = \frac{c_{12}d}{2c_{11}}\varepsilon_c \pm \sqrt{\left(\frac{c_{12}d}{2c_{11}}\varepsilon_c\right)^2 - \left(c_{12}\lambda + 3\left(c_{32} - \frac{c_{12}c_{31}}{c_{11}}\right) q_1^2 + 2\varepsilon_c c_{22} q_1\right)} \tag{23}$$

for the pairs λ, q_1 satisfying *Eq.* (20).

Bifurcation parameter λ appears only under the square-root (23), thus the loss of stability is a saddle-focus bifurcation. The sign of the eigenvalues depends only on the second term of the expression under the square-root. If it is negative, both of the two eigenvalues have the sign of $\frac{c_{12}d}{2c_{11}}\varepsilon_c$. For the data of [20], it is negative, thus the equilibrium is a stable focus point. If the second term of the expression under the square-root is positive, one of the eigenvalues will be positive, while the other one is negative, thus the equilibrium is an unstable saddle point.

To calculate the analytical solution $\lambda = \lambda(q_1)$ and to substitute it into (23) would require a long mathematical investigation. To avoid it, the numerical values of [20] will be used again. The second term of the expression under the square-root in (23) is

$$4.18\lambda - 6.12 q_1^2 + 0.0712 q_1 \,. \tag{24}$$

By expressing λ from the bifurcation equation (21) and substituting into (24), the sign of the important term is the same as that of

$$-q_1^2 + 0.0851 q_1 - 0.0395 \frac{1}{q_1} \,. \tag{25}$$

After calculating the values of (25), the stability properties in *Fig. 2* are obtained.

4. The Case of an Eccentric Orbit

The results of the analytical investigations show that the number of the equilibria of the inplane attitude motion of the dumbbell satellite flying on a low orbit where a weak atmospheric drag can be taken into consideration changes if the length of the dumbbell satellite is varied. For a very large length the number of the possible equilibria is four. There exists a critical length at which the number is three and for shorter ones only two equilibria exist. Near to the critical length on quasi-static retrieving of the tether, two equilibria are coming together. One of them is an asymptotically stable focus while the other one is an unstable saddle. After a saddle-focus bifurcation both of them disappear, meanwhile a third asymptotically stable equilibrium enters the vicinity of the bifurcation point. This third one does not have any bifurcation, remains asymptotically stable and persists.

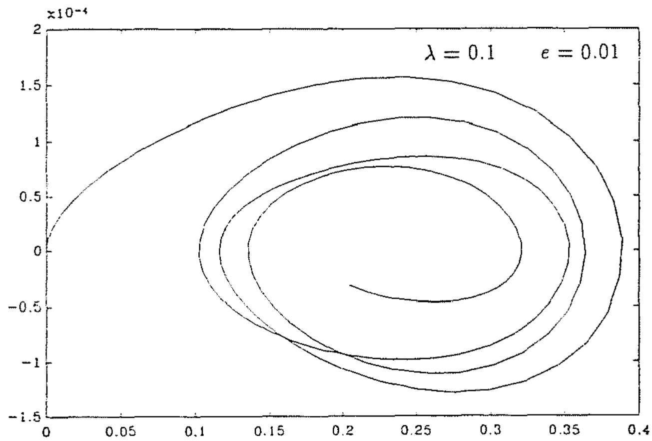

Fig. 3.

From the investigation into a circular orbit, the following question arises. What happens if the satellite has some excitation being obviously present for an eccentric orbit. Generally, one may hope in getting some local oscillatory motion around the stable equilibria of the unperturbed (circular orbit) system. The problem is the appearance of the saddle-focus bifurcation of one of them. In the following, some phase plots are showed as results of numerical simulations of *Eq.* (15) for various values of eccentricity, and the normalized dimensionless length previously called the bifurcation parameter λ.

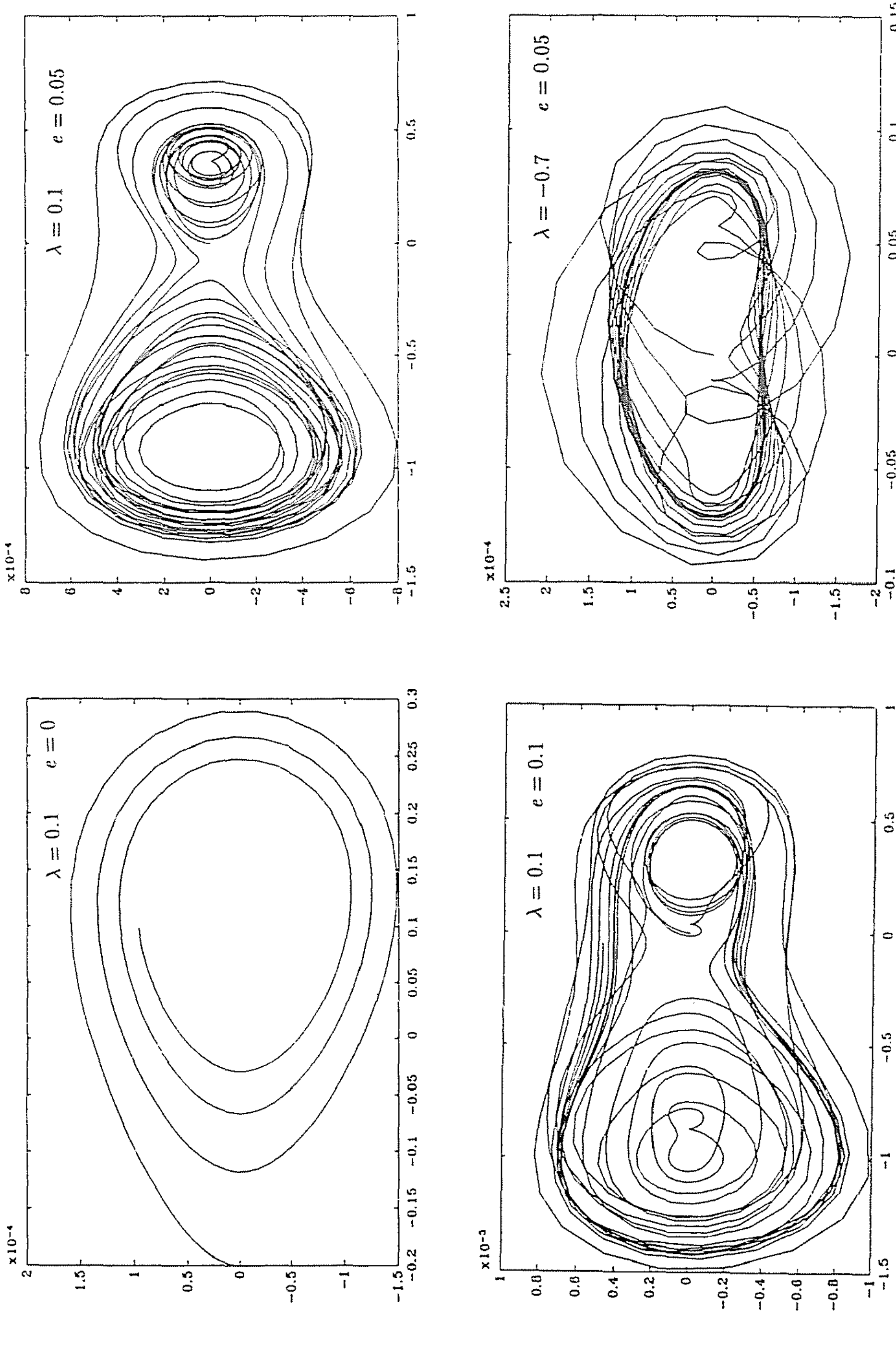

Fig. 4.

For the simulations, a fixed step Runge-Kutta differential equation solver routine of the MATLAB package was applied. In the following few figures, these phase plots are presented. The calculations are performed for both circular and eccentric orbits. Some figures show cases when, for circular orbit, the number of the equilibria is three (supercritical cases) and some others concern the one equilibrium (subcritical) situations.

5. Summary

The results of the investigation into a dumbbell satellite moving on circular and eccentric LEO show that both eccentricity and the distance of the two parts of the system has important effect on the motion of it. The quasi-static variation of the length can change the number of the equilibria of the attitude motion for a circular orbit. The way to do it is called in applied mathematics a saddle-focus bifurcation at a critical length.

The effect of the orbit eccentricity causes an excitation because of the presence of aerodynamical forces on a LEO. As the numerical simulations show for cases when $e = 0.01$, the qualitative picture of the phase plane is very similar to the picture of a circular orbit at the same length.

If the eccentricity is about 0.1 or larger, the excitation plays the main role in the motion. However, for some values of the length, the phase plot has some similarity with the well-known chaotic motion of an excited double-well potential system [17].

In the case of an intermediate eccentricity $e = 0.05$, the similarity is more obvious. For such an eccentricity, some simulations were performed at various values of the length. The regular case (when the length is far enough from the critical one) gives a chaotic motion on a strange attractor mentioned previously [17]. When the length is a little bit less than its critical value, the phase plot also gives the impression of being some kind of strange attractor, but obviously not of the same type.

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