PERIODICA POLYTECHNICA SER. TRANSP. ENG. VOL. 26, NO. 1-2, PP. 61-70 (1998)

# CRITICAL EIGENMODES AND INTRINSIC MATERIAL LENGTH

Péter B. BÉDA

Research Group of the Dynamics of Machines and Vehicles
Department of Chassis and Lightweight Structures
Technical University of Budapest
P.O. Box 91. H-1502 Budapest, Hungary
Phone: (36) 1-463-1755
e-mail: bedap@kme.bme.hu

Received: July 1, 1998

## Abstract

The main aim of the paper is to study how the inclusion of nonlocality (gradient dependent terms) into the constitutive equations changes the mathematical description of material instability problems. The motivation comes from the theory of dynamical systems, when the stability analysis can be performed by finding eigenvalues and eigenvectors of certain linear operators. Some of the eigenvectors define critical eigenmodes and the postbifurcation investigation is based on these critical eigenmodes. The results show that by considering the body as a dynamical system the critical eigenmodes can be selected when the constitutive equation contains the second gradient term. The wavelength of the dominant eigenmode can be identified with the intrinsic material length.

*Keywords:* material instability, dynamical systems.

## 1. Introduction

In the theory of dynamical systems [15] the loss of stability of a state of the system means that the real part of some eigenvalues of an operator describs its behavior changes sign. The eigenvectors connected to them are called the critical eigenmodes [13]. In nonlinear case, the postbifurcation can be studied and described analytically by using these critical eigenmodes. The aim of this paper is to study postlocalization in a similar way by considering solid continua as dynamical systems [1], [3]. This kind of investigation is closely related to the perturbation analysis [6], [16].

Unfortunately, for the classical setting [10] there is no possibility to obtain specific critical eigenmodes at the onset of material instability. On the other hand, in the finite element calculation of material instability problems the classical formulation of the basic equations of solid continua results in a definite mesh dependence [4], [11], [12]. These are very similar phenomena. In those papers the mesh dependence was eliminated by the inclusion of rate dependence or nonlocality (gradient effects) into the constitutive equations.

In both cases an intrinsic length appeared for dynamic or static instability problems.

In this paper we study how the inclusion of gradient dependent terms into the constitutive equations leads to the appearance of critical eigenmodes and what kind of mathematical interpretation can be attached to the intrinsic length.

The second part presents the basic equations for the solid body. These equations will be transformed into the velocity field, because such form is convenient for the following investigation. The third part introduces the basic notions of the dynamical system theory and formulates a stability condition for a state of the body based on the Lyapunov stability definition. In part four a one dimensional linear problem is treated and part five shows how this eigenmode can be used for the description of the nontrivial (post-bifurcated) state. The wavelength of this eigenmode can be identified with the intrinsic length.

## 2. The Basic Equations

First the basic equations are studied. In the case of small strain the kinematic equation is

$$\epsilon = \frac{1}{2}\left(\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}\right), \tag{1}$$

where $\epsilon$ is the strain tensor, $\mathbf{u}$ is the displacement vector and $\circ$ denotes diadic product. The equation of motion without body forces is

$$\rho \ddot{\mathbf{u}} = \mathbf{T}\nabla, \tag{2}$$

where $\rho$ is the mass density and $\mathbf{T}$ denotes the symmetric Cauchy stress tensor.

Let the constitutive equation have the rate form [14]

$$\dot{\sigma} = F\left(\dot{\epsilon},\ \ddot{\epsilon},\ \nabla^2\dot{\epsilon}\right) \tag{3}$$

representing both rate-dependent and nonlocal effects.

By studying the stability of a state $S^0$ described by $\sigma^0$, $\epsilon^0$ ...the constitutive equation can be linearized at $S^0$. Let the new variables $\bar{\sigma} = \sigma - \sigma^0$, $\bar{\epsilon} = \epsilon - \epsilon^0$ ...be introduced for the perturbations. Because at state $S^0$ the variables satisfy (3) the linearized rate constitutive equation is [14]

$$\dot{\bar{\sigma}} = \mathbf{C}^1\dot{\bar{\epsilon}} + \mathbf{C}^2\ddot{\bar{\epsilon}} + \mathbf{C}^3\nabla^2\dot{\bar{\epsilon}}. \tag{4}$$

Now *Eqs* (1), (2) and (4) are the basic equations for the stability investigation of state $S^0$. These equations should be transformed into the velocity field $\mathbf{v}$. For the sake of simplicity the bars are omitted in the following but all equations concern the small perturbations of the state $S^0$, thus all the calculations are performed in a sufficiently small neighbourhood of $S^0$.

For small strains $\mathbf{T} = \sigma$, then from (2), (4) and by using the rate form of (1)

$$\begin{aligned} 2\rho\ddot{\mathbf{v}} &= \mathbf{C}^1 (\mathbf{v} \circ \nabla + \nabla \circ \mathbf{v}) \nabla + \mathbf{C}^2 (\dot{\mathbf{v}} \circ \nabla + \nabla \circ \dot{\mathbf{v}}) \nabla \\ &+ \mathbf{C}^3 \nabla^2 (\mathbf{v} \circ \nabla + \nabla \circ \mathbf{v}) \nabla. \end{aligned} \tag{5}$$

In the following the stability investigation of state $S^0$ will be based on *Eq.* (5).

## 3. Dynamical Systems, Static and Dynamic Bifurcations

In abstract form (5) reads

$$\ddot{v} = F^1 v + F^2 \dot{v}. \tag{6}$$

Here $v = (v_1, v_2, v_3)$ is a vector of the co-ordinates of the velocity field satisfying the boundary conditions and $F^1$, $F^2$ are linear differential operators defined by the right hand side of (5). *Eq.* (6) can be considered as an infinite dimensional dynamical system.

The stability of a state of the continuum ($S^0$ for example) is defined by the Liapunov stability of a solution $v^0(t)$ of (6). That is, a state represented by $v^0(t)$ is stable, when the perturbed velocity field $v^0(t) + \bar{v}(t)$ remains sufficiently close to the unperturbed one. Such definitions are also used in solid mechanics [7], [8], [9]. The stability investigation of the solution $v^0(t)$ starts with a transformation into a local form at that solution by substituting

$$v(t) = v^0(t) + \bar{v}(t)$$

into (6),

$$\ddot{v}^0 + \ddot{\bar{v}} = F^1 \left(v^0 + \bar{v}\right) + F^2 \left(\dot{v}^0 + \dot{\bar{v}}\right). \tag{7}$$

While $v^0$ is a solution of (6) and $F^1$, $F^2$ are linear operators, the first terms of each part in (7) are equal, thus the equation of motion (7) of the perturbation $\bar{v}(t)$ has the same form as (6). Then (7) should be transformed into a system of first order equations by introducing new variables

$$y_1 = \bar{v}_1, \quad \ldots, \quad y_3 = \bar{v}_3, \qquad y_4 = \dot{\bar{v}}_1, \quad \ldots, \quad y_6 = \dot{\bar{v}}_3,$$

and vectors

$$y_\varphi, \quad (\varphi = 1, \ldots, 3), \qquad y_\psi, \quad (\psi = 4, \ldots, 6).$$

The transformed equations are

$$\dot{y}_{\varphi} = y_{\psi}, \tag{8}$$

$$\dot{y}_{\psi} = F^1 y_{\varphi} + F^2 y_{\psi}. \tag{9}$$

Now the stability properties are determined by the eigenvalues of the linear operator $\widehat{F}$ defined by the right hand sides of (8) and (9),

$$\widehat{F}\,(y_{\varphi}, y_{\psi}) = \left(y_{\psi}, F^1 y_{\varphi} + F^2 y_{\psi}\right).$$

By using Liapunov's indirect method [5] the solution $v^0$ is asymptotically stable, when the real parts of all eigenvalues of $\widehat{F}$ are negative. In case of zero real parts, the system is on the stability boundary. The characteristic equation of $\widehat{F}$ reads

$$\begin{aligned} \lambda y_{\varphi} &= y_{\psi}, \\ \lambda y_{\psi} &= F^1 y_{\varphi} + F^2 y_{\psi}. \end{aligned} \tag{10}$$

By substituting the first group of (10) into the second group

$$\lambda^2 y_{\varphi} - \lambda F^2 y_{\varphi} - F^1 y_{\varphi} = 0 \tag{11}$$

is obtained. The condition of stability is $\mathrm{Re}\lambda_i \leq 0$, $i = 1 \ldots$ for all $\lambda_i$ satisfying (11). The typical way in which stability is lost is by: (a) a real $\lambda_c$ or (b) the real part of a pair of complex conjugate $\lambda_{c1}$ and $\lambda_{c2} (= \bar{\lambda}_{c1})$ changes sign, while all the others satisfy $\mathrm{Re}\lambda_i < 0$, $i \neq c$ or $i \neq c1,\ c2$, respectively. Thus the loss of stability can either be a generic static (a) or dynamic (b) bifurcation [3]. This classification is quite similar to the one in [16], where a) is called the strain hardening type and b) is the rate sensitivity type. In case a) (11) has a (real) eigenvalue $\lambda_c = 0$. Then the condition of the static bifurcation is

$$F^1 y_{\varphi} = 0. \tag{12}$$

Note that this phenomenon is the same as the divergence instability or the onset of strain localization [10]. In this case also the uniqueness of the solution $v^0$ is lost and other, nontrivial solutions can appear.

At the dynamic bifurcation (b) the eigenvalues are imaginary conjugates, $\lambda_{c2} = \bar{\lambda}_{c1}$, thus the condition is

$$F^2 y_{\varphi} = 0. \tag{13}$$

The main difference between the cases a) and b) is that in b) the uniqueness of $v^0$ persists, but the Liapunov stability is lost.

In the following the static bifurcation of a stationary (or steady state) solution will be studied. A solution of (8) and (9) is called stationary, when

$$\dot{y}_{\varphi} = \dot{y}_{\psi} = 0.$$

Let us study what happens with such solutions at a static bifurcation. In the investigation also the nonlinear terms $\widehat{N}(y_\varphi, y_\psi)$ are necessary. When the nonlinear terms are included from (8) and (9) for the stationary solutions

$$0 = F^1 y_\varphi + N(y_\varphi) \tag{14}$$

is obtained, where $N(y_\varphi) = \widehat{N}(y_\varphi, 0)$. Assume that $F^1$ depends on a (for example loading) parameter $\mu$ and at $\mu = 0$ condition (12) of the static bifurcation is satisfied with the eigenmode $y_\varphi^0$,

$$F^1\Big|_{\mu=0} y_\varphi^0 = 0. \tag{15}$$

Defining $\bar{F}^1(\mu) = F^1 - F^1|_{\mu=0}$ $Eq.$ (14) is

$$0 = \left(\bar{F}^1(\mu) + F^1|_{\mu=0}\right) y_\varphi + N(y_\varphi). \tag{16}$$

In static bifurcation theory [2] in a small neighbourhood of $v^0$ (or state $S^0$) the nontrivial solution can be searched for in the form

$$y_\varphi = q y_\varphi^0,$$

where $q$ is a small real number. By substituting into (16)

$$0 = q\bar{F}^1(\mu) y_\varphi^0 + N\left(q y_\varphi^0\right) \tag{17}$$

is obtained because of (15). By introducing a scalar product $< \ldots >$ from (17) an approach of the bifurcation equation [13]

$$0 = q\left\langle y_\varphi^0, \bar{F}^1(\mu) y_\varphi^0\right\rangle + \left\langle y_\varphi^0, N\left(q y_\varphi^0\right)\right\rangle \tag{18}$$

is obtained, which is a non-linear algebraic equation for $q$. By performing power series expansions and considering only the first few terms it can be solved for $q = q(\mu)$. Then for sufficiently small $\mu$ the nontrivial solution

$$y_\varphi = q(\mu) y_\varphi^0$$

is obtained.

In the next parts the static bifurcation of a solid body will be studied in a one dimensional problem.

## 4. The One Dimensional Linear Case

Let a rod of length $L$ be considered. Then the constitutive equation in rate form is

$$\dot{\sigma} = c_1\dot{\epsilon} + c_2\ddot{\epsilon} - c_3\frac{\partial^2\dot{\epsilon}}{\partial x^2}.$$

*Eqs* (8) and (9) in this case are

$$\dot{y}_1 = y_2, \tag{19}$$

$$\dot{y}_2 = \left(c_1\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right) y_1 + c_2\frac{\partial^2}{\partial x^2}y_2. \tag{20}$$

Now the characteristic equation (11) is

$$\lambda^2 y_1 - \lambda c_2\frac{\partial^2}{\partial x^2}y_1 - \left(c_1\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right) y_1 = 0. \tag{21}$$

In case of homogeneous boundary conditions

$$v_1 = e^{i\alpha_k x}, \qquad \text{where} \qquad \alpha_k = \frac{k\pi}{L} \quad (k = 1, \ldots) \tag{22}$$

should be substituted into (21) and for the eigenvalues

$$\lambda_{1,2,k} = \frac{-c_2\alpha_k^2 \pm \sqrt{c_2^2\alpha_k^4 - 4\alpha_k^2\left(c_3\alpha_k^2 + c_1\right)}}{2}. \tag{23}$$

When $c_1$ is positive, all the eigenvalues are negative, thus the state is stable. The a) type loss of stability happens, when

$$\left(c_3\alpha_k^2 + c_1\right) = 0. \tag{24}$$

Thus at

$$\alpha_k = \alpha_{k*} = \sqrt{-\frac{c_1}{c_3}}. \tag{25}$$

In this case the only function of form (22) satisfying (25) is

$$v = e^{i\sqrt{-\frac{c_1}{c_3}}x}. \tag{26}$$

Obviously, when for some $k$

$$\alpha_k > \alpha_{k*}, \tag{27}$$

one of the eigenvalues $\lambda_{1,2,k}$ has a positive real part. For that $k$ this implies instability.

While $c_1$ is the tangent of the stress-strain diagram at state $S^0$ and during a quasistatic loading process it gets increasingly negative values on the softening side, the first critical $\alpha_k$ is at $k = 1$. For a fixed $c_1 < c_{1crit} = -\alpha_1^2 c_3$ in the instability part of the stress-strain diagram several unstable eigenvalues could exist. Then in the unstable zone, all perturbations with wavelength

$$l > 2\pi\sqrt{-\frac{c_3}{c_1}}$$

lead to unstable behaviour.

Expression (23) shows the differences of the gradient dependent and independent cases. When the material is gradient independent, $c_3 = 0$. Then from (23)

$$\lambda_{1,2,k} = \frac{-c_2\alpha_k^2 \pm \sqrt{c_2^2\alpha_k^4 - 4\alpha_k^2 c_1}}{2}. \tag{28}$$

The condition of the a) type loss of stability is $c_1 = 0$, because then (28) implies

$$\lambda_{1,2,k} = \frac{-c_2\alpha_k^2 \pm |c_2|\,\alpha_k^2}{2}. \tag{29}$$

By comparing (24) and (29) the main difference is that (24) defines a critical $k = k*$ (see (25)) and consequently a critical eigenmode $e^{i\alpha_{k*}x}$ for the perturbation. In (29) all values $k$ $k = 1, 2 \ldots$ and all perturbations $e^{i\alpha_{k*}x}$ are critical whenever $c_1 = 0$. In other words, for gradient independent constitutive equation all wavelengths are critical. At the end of part 3 the nontrivial solutions were searched for as linear combination of the critical eigenmode. Such study cannot be performed for rate independent constitutive equation because of the infinite number of critical eigenmodes.

In case of an adiabatic localization $L$ tends to infinity. Then the instability condition (24) can be satisfied by any arbitrary small real $\alpha$, thus the state loses stability, when $c_1$ gets negative values. Thus the stability boundary for the adiabatic case is $c_1 = 0$. In the softening region for a fixed negative $c_1$ similarly to (25) the wavelength of the critical perturbation

$$l^* = 2\pi\sqrt{-\frac{c_3}{c_1}}$$

can be defined as an intrinsic length. All periodic perturbations having larger wavelength than $l^*$ cause instability. The following part explains how $l^*$ describes the postlocalization when also nonlinearity is present.

## 5. The Effect of Material Nonlinearity

In this section a non-linear constitutive equation proposed by [17] is used

$$\dot{\sigma} = c_1\dot{\epsilon} + c_2\ddot{\epsilon} - c_3\frac{\partial^2\dot{\epsilon}}{\partial x^2} + c_4\left(\frac{\partial\dot{\epsilon}}{\partial x}\right)^2 \tag{30}$$

for the adiabatic postlocalization investigation in the one dimensional case of the previous part. Assume that the loss of stability of state $S^0$ happens at $c_1 = c_{10}$. By introducing a small $0 < \mu \ll 1$

$$c_1 = c_{10} - \mu. \tag{31}$$

By using (30), (31) and the one dimensional form of (1) and (2) the equation of motion for the velocity field is

$$\rho\ddot{v} = \left(c_{10}\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right)v - \mu\frac{\partial^2}{\partial x^2}v + c_2\frac{\partial^2}{\partial x^2}\dot{v} + c_4\left(\frac{\partial^3 v}{\partial x^3}\right)^2. \tag{32}$$

Since the localization is a static bifurcation [3], the postbifurcation investigation can be restricted to the steady state solutions $\ddot{v} = \dot{v} = 0$ of (32). Then instead of (32)

$$\left(c_{10}\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right)v - \mu\frac{\partial^2}{\partial x^2}v + c_4\left(\frac{\partial^3 v}{\partial x^3}\right)^2 = 0 \tag{33}$$

is used.

In the linear study of the previous section the eigensolution (26) was obtained for $v$. Thus, for homogeneous boundary conditions $\sin(\alpha x)$ can be identified as the first critical eigenmode function. As outlined in section 3 the bifurcated nontrivial solution of (33) can be searched for a linear combination of the critical eigenmodes. Now there is a unique critical eigenmode, thus

$$v^c = q\sin(\alpha x),$$

where $|q| \ll 1$ and $\alpha$ satisfies (25). Function $v^c$ can be substituted into (33) and then the scalar product (the left hand side of (18))

$$\begin{aligned} g(q,\mu) &= \int_0^L \left(\left(\left(c_{10}\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4} - \mu\frac{\partial^2}{\partial x^2}\right) q\sin(\alpha x)\right)\sin(\alpha x)\right. \\ &+ \left. c_4\left(\frac{\partial^3 q\sin(\alpha x)}{\partial x^3}\right)^2\right)\sin(\alpha x)\mathrm{d}x \end{aligned}$$

defines function $g(q,\mu)$ for the approximate bifurcation equation

$$g(q,\mu)=0. \tag{34}$$

By solving (34)

$$q=-\frac{3\mu}{4\pi c_4\alpha^4} \tag{35}$$

and thus the nontrivial solution is

$$v_1^c=-\frac{3}{4\pi c_4\alpha^4}\mu\sin(\alpha x),$$

or by using (25)

$$v_1^c=-\frac{3c_3^2}{4\pi c_4 c_1^2}\mu\sin\left(x\sqrt{-\frac{c_1}{c_3}}\right). \tag{36}$$

In (36) the coefficient of $x$ in the sine shows that the intrinsic length being the wavelength of $v^c$ is the wavelength of the nontrivial solution.

## 6. Summary

The effect of the inclusion of gradient dependence on the divergence type material instability was studied. The method of the analysis was based on the theory of dynamical systems. Firstly, a stability condition was formulated for the differential operators defined by the system of basic equations of the solid continuum.

In a one dimensional case a linear problem was treated. When the constitutive equation contains the second gradient term, a unique eigenvalue changes sign at the loss of stability. It makes possible to find a critical eigenmode and, for a non-linear constitutive equation, a non-trivial solution can be obtained. For gradient independent materials such calculation is impossible, because at the loss of stability infinite number of eigenvalues change sign at the same time.

The intrinsic material length could be identified with the minimal wavelength of the unstable perturbations. This is the wavelength of the nontrivial solutions at the static bifurcation.

## Acknowledgement

This work was supported by the National Scientific Research Foundation Hungary (under contract: OTKA F017331). This support is gratefully acknowledged.

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