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RESEARCH ARTICLE

Railway track dynamics with periodically varying stiffness and damping in the Winkler foundation

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Abstract

We investigate the dynamics of the train/track system in case of an inhomogeneous longitudinal subgrade stiffness/damping distribution varying periodically along the track. In our study the track is modelled by a Bernoulli-Euler beam laying on a Winkler foundation of stiffness/damping parameters represented by continuous periodic functions. The damped oscillatory load is moving along the track at a constant velocity. In order to obtain an analytical solution to the boundary problem we utilize our previous method for the compactly supported case, based on the approximation of the parameter functions by generalized functions. A finite closed-form formula can be obtained with the help of principal values in the sense of Cauchy.

Keywords

railway track dynamics · beam equation · inhomogeneous supporting field

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1 Introduction

It is a long-standing problem of track/vehicle system dynamics to describe the motion forms of loads moving along beams supported by subgrades of varying stiffness/damping parameters, see e.g. [2].

In the present paper we investigate the simple system consisting of a moving, damped oscillatory load, a Bernoulli-Euler beam and an elastic subgrade of continuous, periodically varying stiffness/damping parameters.

By using the method of [3], solving similar problems for compactly supported continuous foundation stiffness/damping functions, we obtain the analytical, closed-form solution to our problem with the help of principal values in the sense of Cauchy.

2 System model

In our model we consider a damped oscillatory load $G\exp(wt)$ moving along a Bernoulli-Euler beam at a constant velocity v, where $w = \alpha + i\omega$ is the complex frequency of the load: in case $\alpha = 0$ we have a harmonic load, while for $w = 0$ the load is constant.

Let EI and ρA be the usual parameters of the beam, which is laying on an elastic Winkler foundation of continuously varying, L-periodic stiffness and damping parameters

$$s_0 + s(x) \quad \text{and} \quad k_0 + k(x),$$

respectively. Here s_0 and k_0 are the average stiffness and damping of the foundation, while s and k are periodic continuous functions with (minimal) period $L$and with average 0, i.e.

$$\int_0^L s(y)\mathrm{d}y = 0 \quad \text{and} \quad \int_0^L k(y)\mathrm{d}y = 0$$

are satisfied.

The motion of the system is governed by the Bernoulli-Euler partial differential equation

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + (k_0 + k(x))\frac{\partial z}{\partial t} + (s_0 + s(x))z = G\exp(wt)\delta(x - vt)$$

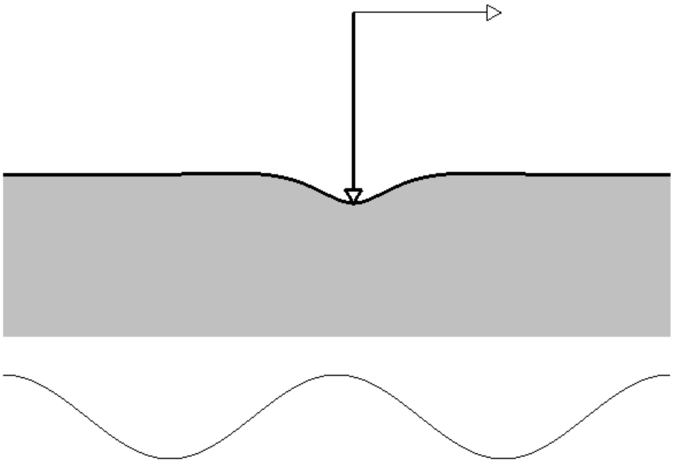

Fig. 1. The track/vehicle system with periodic continuous foundation parameters

of varying coefficients, with damped oscillatory excitation along the curve $x = vt$.

The above partial differential equation must satify boundary condition

$$\lim_{|x|\to\infty} z(x,t) = 0.$$

3 Approximate boundary problem with compact supports

In paper [3] the similar problem of continuous foundation parameters has been solved in the case, when the continuos subgrade parameters have compact supports, i. e. when in partial differential equation

$$EI\frac{\partial^4 z}{\partial x^4} + \rho A\frac{\partial^2 z}{\partial t^2} + (k_0 + k(x))\frac{\partial z}{\partial t} +$$
$$(s_0 + s(x))z = G\exp(wt)\delta(x - vt)$$

the functions k and s are continuous functions on the finite interval $[x_0, y_0]$, and vanish outside.

For the solution of this auxiliary problem one can use the characteristic polynomial

$$P(\lambda) = EI\lambda^4 + \rho A v^2\lambda^2 -$$

$$v(k_0 + 2\rho A w)\lambda + (s_0 + k_0 w + \rho A w^2),$$

of the differential equation, investigated e.g. in [1].

Let λ_i denote the roots of the characteristic polynomial above, and we define sign

$$\sigma_i := -sgn(Re\lambda_i)$$

for $i = 1, ..., 4$, cf. [3].

If we introduce auxiliary functions

$$c_i(x) := \frac{\sigma_i}{4EI\lambda_i^3}(s(x) + (w - \lambda_i v)k(x)), \quad i = 1, \ldots, 4,$$

then, with the help of the approximation of discrete functions by generalized functions (cf. [3]) the solution to the compactly supported problem can be written into integral form

$$u(x,t) = G\sum_{i=1}^{4}\frac{\sigma_i}{P'(\lambda_i)}$$

$$\exp\left(wt + \lambda_i(x - vt) - \int_{l_i}^{x} c_i(y)dy\right) H(\sigma_i(x - vt))$$

with $l_i := \begin{cases} x_0 \text{ if } \mathrm{Re}\lambda_i < 0, \\ y_0 \text{ if } \mathrm{Re}\lambda_i > 0. \end{cases}$

4 Transition to the periodic case

In order to generalize our results to the periodic foundation case we intend to use the principal values in the sense of Cauchy.

Since continuous functions s and k can take their zeroes at different points, we are looking for a place x_0, where both functions have relatively small values.

Let x_0 be a point on the real line, where function

$$|s(x) + wk(x)|$$

is minimal. (Continuity of functions s and k implies the existence of such a point.)

At first we compute the solution to the problem in case of a support, where our parameters vary only inside the finite interval

$$[x_0 - nL, x_0 + nL]$$

of length $2nL$. Here n is a natural number and L stands for the common period of functions s and k.

The results of [3], mentioned in the previous section, imply, that the solution u^n to this case has the form

$$u^n(x,t) = G\sum_{i=1}^{4}\frac{\sigma_i}{P'(\lambda_i)}$$

$$\exp\left(wt + \lambda_i(x - vt) - \int_{x_0-\sigma_i nL}^{x} c_i(y)H(nL - |y - x_0|)\mathrm{d}y\right)$$

$$\mathrm{H}(\sigma_i(x - vt)).$$

The integral in the above formula can be transformed into the form

$$\int_{x_0}^{x} c_i(y)H(nL - |y - x_0|)\mathrm{d}y = \int_{x_0}^{x} c_i(y)\mathrm{d}y\ H(nL - |x - x_0|),$$

since if for any y, settled between x_0 and x, relation $|y - x_0| < nL$ holds, then it is equivalent to the satisfaction of relation $|x - x_0| < nL$.

Functions $c_i,\ i = 1, ..., 4$ have vanishing averages:

$$\int_{x_0-L}^{x_0} c_i(y)\mathrm{d}y = 0,$$

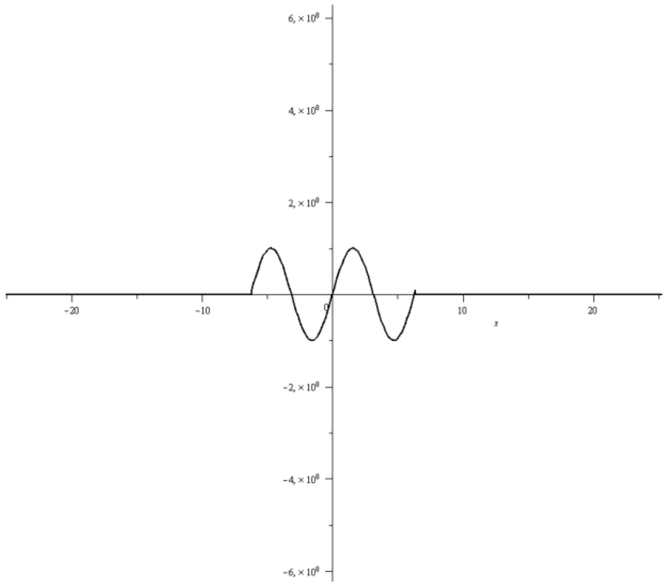

Fig. 2. The shape of the foundation parameters in approximation step n=1

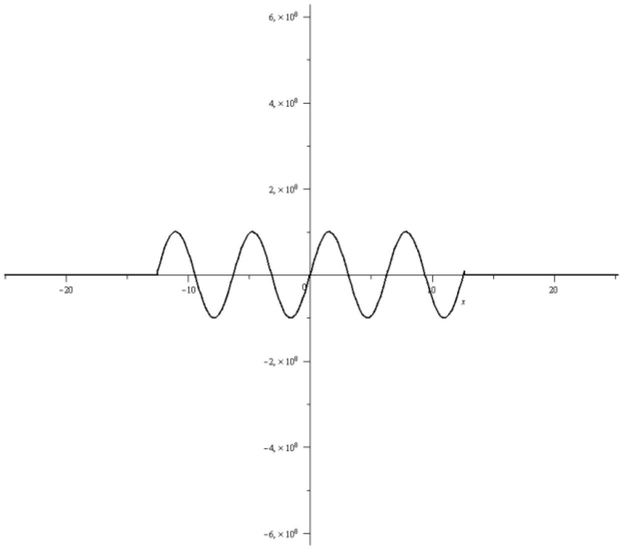

Fig. 3. The shape of the foundation parameters in approximation step $n = 2$

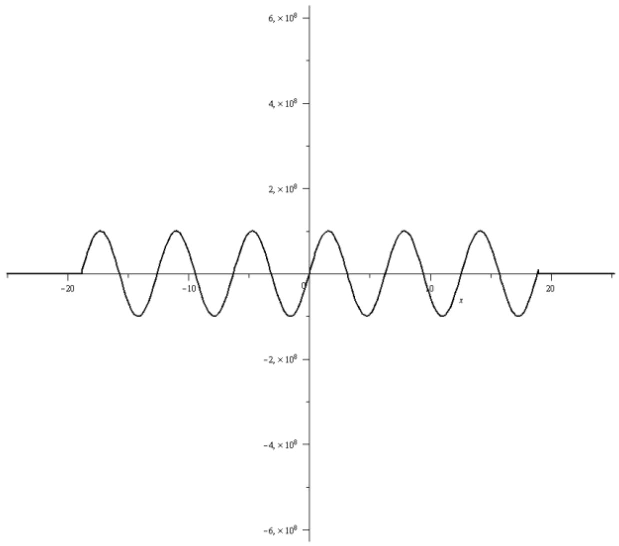

Fig. 4. The shape of the foundation parameters in approximation step $n = 3$

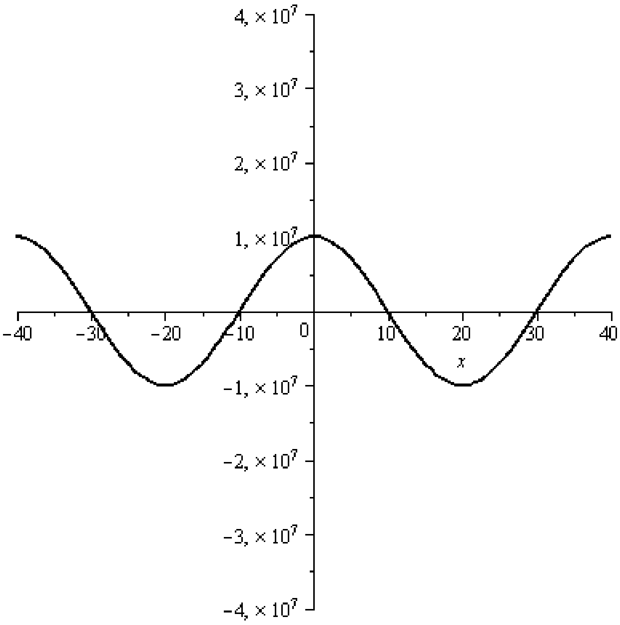

Fig. 5. The shape of the foundation parameters

hence the periodicity of c_i implies

$$\int_{x_0-\sigma_i nL}^{x} c_i(y)\mathrm{d}y = \int_{x_0}^{x} c_i(y)\mathrm{d}y,$$

and for the solution to this case

$$u^n(x,t) = G\sum_{i=1}^{4}\frac{\sigma_i}{P'(\lambda_i)}$$

$$\exp\left(wt + \lambda_i(x - vt) - \int_{x_0}^{x} c_i(y)\,\mathrm{d}y\ \mathrm{H}(nL - |x - x_0|)\right)$$

$$H(\sigma_i(x - vt))$$

is satisfied.

The solution to the original, continuously supported problem can now be given as limit

$$u(x,t) = \lim_{n\to+\infty} u^n(x,t).$$

Since

$$\lim_{n\to+\infty}\mathrm{H}(nL - |x - x_0|) = 1$$

holds, i.e. $|x - x_0| < nL$ is satisfied for sufficiently large numbers n, for the solution we have formula

$$u(x,t) = G\sum_{i=1}^{4}\frac{\sigma_i}{P'(\lambda_i)}$$

$$\exp\left(wt + \lambda_i(x - vt) - \int_{x_0}^{x} c_i(y)\mathrm{d}y\right) H(\sigma_i(x - vt)),$$

where

$$c_i(x) := \frac{\sigma_i}{4EI\lambda_i^3}(s(x) + (w - \lambda_i v)k(x)), \quad i = 1, ..., 4$$

is satisfied and

$$|s(x_0) + wk(x_0)|$$

is minimal.

5 Numerical results

In our simulation we use beam data $EI = 6 \cdot 10^6$ Nm^2 and $\rho A = 60$ kg/m. The constant load is $G = 6.5 \cdot 10^4$ N, moving along the beam at velocity $\upsilon = 40$ m/s.

The average values of the foundation stiffness and damping are s_0 =9.05·10^7 N/m^2 and k_0 =47 250 Ns/m^2, respectively, while the continuously varying, averageless stiffness and damping is respresented by functions

$$s(x) = \cos(\pi x/(20\text{ m})) \cdot 10^7\text{N/m}^2$$

$$k(x) = \cos(\pi x/20(\text{ m}) \cdot 2500\text{ Ns/m}^2$$

of period L=40 m.

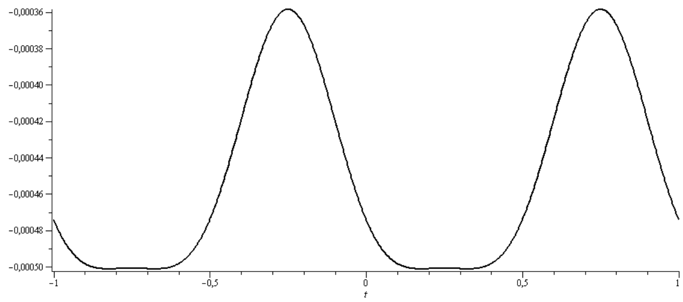

Fig. 6. The motion of the load

Fig. 6 illustrates the motion of the constant load moving along the beam laying on an elastic subgrade of periodically varying, continuous parameter functions of the form shown in Fig. 5.

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