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RESEARCH ARTICLE

Application of Principle of Work and Energy in Its Differential Form to MDOF Systems

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Abstract

This paper is intended to reveal some new aspects of the principle of work and energy in its differential form. A theorem is given to state the principle of work and energy in differential form that can be applied alone to build up all the dynamical equations of a special class of MDOF mechanical systems. Some illustrations and discussions are presented to demonstrate the refined applicability of the principle of work and energy in its differential form through the comparison with the Lagrange equation method.

Keywords

kinetic energy theorem, applicability, limitation, MDOF system, Lagrange equation, generalized coordinates

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1 Introduction

As well known, the principle of work and energy for a particle system states that the change in the kinetic energy of a particle system between the initial instant and end instant is equal to the work done by all the internal forces and the resultant external forces acting on the system during this time interval, and usually is formulated as

$$T_2 - T_1 = W_i + W_e \tag{1}$$

where T_1 and T_2 represent the kinetic energy of the initial instant and the end instant, respectively, and W_i and W_e represent the work done by the internal forces and resultant external forces acting on the system, respectively.

Indeed, Eq. (1) is the principle of work and energy of integral form, and there also exists in its differential form, which is formulated as

$$\mathrm{d}T = \delta W$$

where $\mathrm{d}T$ is the differential of the kinetic energy, and δW represents the elementary work done by inner forces and the resultant external forces.

The principle of work and energy of integral form is frequently used to derive the first integral of the dynamical system, which is sometimes related to the conservation principle of mechanical systems; while the principle of work and energy in differential form is rarely used independently, and most of the illustrating examples of its application involve only single-degree-of-freedom (SDOF) systems. This is essentially because for the multi-degree-of-freedom (MDOF) system, the differential of kinetic energy is the linear combination of small real displacements that are not independent of each other, which means the front coefficient terms cannot vanish. Therefore, in general we have to resort to the Lagrange equation method to deal with MDOF systems.

In this study, we will gain an insight into the principle of work and energy in differential form and present a new result on its application. The key point is that by comparing with the Lagrange equation, we find a suitable restrictive condition on the system under which all the dynamical equations of the mechanical system can be derived by applying only the differential form of the principle of work and energy alone. Examples which do not satisfy this restrictive condition are also presented to illustrate the limitation of this method and the necessity of the restrictive condition on the system.

The article is organized as follows. In Section 2, a theorem is given to show that under certain restrictive conditions for a mechanical system, all its dynamical equations can be derived from only the differential form of principle of work and energy alone, with the application to a practical problem presented as an illustrative example. Section 3 carries out an in-depth discussion on a tricky case and a total failure case of the application of this method in the absence of the restrictive conditions.

2 The condition for application of kinetick theorem of energy of differential form to MDOF systems

Theorem1. For a MDOF system with bilateral, scleronomic, holonomic and ideal constraints, if the expression of kinetic energy does not contain the position variables of the generalized coordinates, then all the dynamical equations of the system can be derived only from the differential form of the principle of the work and energy alone, i.e. by collecting the items in the equation according to differential generalized coordinate items and then vanishing the front coefficients.

Proof: Suppose a MDOF system is composed of n particles and is of k degrees of freedom. Let $q_1, q_2, \cdots, q_k$ be the generalized coordinates of the system. Then, every particle's position can be expressed as

$$\boldsymbol{r}_i = \boldsymbol{r}_i\left(q_1, q_2, \cdots q_k\right), i = 1, 2, \cdots n \tag{2}$$

Thus, the velocity $\dot{\boldsymbol{r}}_i$ and the kinetic energy of the system T can be expressed respectively as

$$\dot{\boldsymbol{r}}_i = \sum_{j=1}^{k} \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j} \dot{q}_j, \quad i = 1, 2, \cdots n \tag{3}$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i = \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(\sum_{i=1}^{n} m_i \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}\cdot\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_l}\right)\dot{q}_j \dot{q}_l \tag{4}$$

Denoting $m_{jl} = \sum_{i=1}^{n} m_i \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}\cdot\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_l}$, then according to the given condition in the theorem, m_{jl} is constant, i.e. m_{jl} does not contain any generalized coordinates, generalized velocities and time variable. Meanwhile, it is easy to check that $m_{jl} = m_{lj}$. Furthermore, we have

$$\frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2}\sum_{j=1}^{k}\sum_{l=1}^{k} m_{jl}\frac{\partial}{\partial \dot{q}_i}\left(\dot{q}_j\dot{q}_l\right) = \frac{1}{2}\sum_{j=1}^{k}\sum_{l=1}^{k} m_{jl}\left(\frac{\partial \dot{q}_j}{\partial \dot{q}_i}\dot{q}_l + \frac{\partial \dot{q}_l}{\partial \dot{q}_i}\dot{q}_j\right)$$

$$\frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2}\sum_{j=1}^{k}\sum_{l=1}^{k} m_{jl}\left(\dot{q}_l\delta_{ij} + \dot{q}_j\delta_{jl}\right) = \sum_{j=1}^{k} m_{jl}\dot{q}_j$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{k} m_{jl}\dot{q}_j\right) = \sum_{j=1}^{k} m_{jl}\ddot{q}_j, \quad \frac{\partial T}{\partial q_i} = 0 \tag{5}$$

Denote the generalized force by $Q = \sum_{j=1}^{n} \boldsymbol{F}_j \cdot \frac{\partial \boldsymbol{r}_j}{\partial q_i}$, and then the dynamical equations are given by Lagrange equation of 2nd class as following

$$\sum_{j=1}^{k} m_{jl}\ddot{q}_j - Q_i = 0 \qquad \left(i = 1,\ 2,\ \cdots,\ k\right) \tag{6}$$

On the other hand,

$$\begin{aligned}\mathrm{d}T &= \sum_{j=1}^{k}\frac{\partial T}{\partial \dot{q}_j}\mathrm{d}\dot{q}_j = \sum_{j=1}^{k}\left(\sum_{i=1}^{k} m_{jl}\dot{q}_i\right)\mathrm{d}\dot{q}_j \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} m_{jl}\dot{q}_i\ddot{q}_j\mathrm{d}t = \sum_{i=1}^{k}\sum_{j=1}^{k} m_{jl}\ddot{q}_j\mathrm{d}q_i\end{aligned}$$

$$\sum \delta W_i = \sum_{i=1}^{n}\boldsymbol{F}_i\cdot\mathrm{d}\boldsymbol{r}_i = \sum_{j=1}^{k} Q_j\mathrm{d}q_j$$

Employing the differential form of principle of work and energy, the following relationships can be obtained.

$$\sum_{i=1}^{k}\sum_{j=1}^{k} m_{jl}\ddot{q}_j\mathrm{d}q_i = \sum_{j=1}^{k} Q_j\mathrm{d}q_i$$

Namely

$$\sum_{i=1}^{k}\left(\sum_{j=1}^{k} m_{ij}\ddot{q}_j - Q_i\right)\mathrm{d}q_i = 0 \tag{7}$$

Letting the coefficients of the differential generalized coordinate items be zero will straightforwardly lead to

$$\sum_{j=1}^{k} m_{ij}\ddot{q}_j - Q_i = 0 \qquad \left(i = 1,\ 2,\ \cdots,\ k\right), \tag{8}$$

which is the same as the result obtained by Lagrange equation method. #

Theorem 1 reveals that for a special class of MDOF system, the principle of work and energy of differential form can be applied alone to derive all the dynamical equations of the system. Thus, this theorem extends the application of the principle of work and energy. But, it must be pointed out that letting the coefficients terms of the differential of the generalized coordinates be zero doesn't mean that $\mathrm{d}q_i$ is independent, and this is just a formal technique.

Example 1. (Human balance model [5]) Homogeneous bars OB and CD are linked smoothly as shown in Fig. 1, where bar OB represents the human body, and bar CD represents the two arms. Determine the differential equations of the system. $\overline{OA}=l_0$, $\overline{OB}=l_1$, $\overline{CD}=l_2$.

Solution: The constraints of the system are obviously bilateral, scleronomic, holonomic and ideal constraints. If the rotating angle φ of bar OB and the relative rotating angle ψ are chosen as generalized coordinates of the system, then the kinetic energy of the system reads

$$\begin{aligned} T &= \frac{1}{2}J_O\dot{\varphi}^2+\frac{1}{2}m_2l_0^2\dot{\varphi}^2+\frac{1}{2}J_A\left(\dot{\varphi}+\dot{\psi}\right)^2 \\ &= \left(\frac{1}{6}m_1l_1^2+\frac{1}{2}m_2l_0^2\right)\dot{\varphi}^2+\frac{1}{24}m_2l_2^2\left(\dot{\varphi}^2+\dot{\psi}^2+2\dot{\varphi}\dot{\psi}\right) \end{aligned}$$

$$\begin{aligned} \mathrm{d}T &= \left(\frac{1}{3}m_1l_1^2+m_2l_0^2+\frac{1}{12}m_2l_2^2\right)\ddot{\varphi}\mathrm{d}\varphi \\ &+\frac{1}{12}m_2l_2^2\ddot{\psi}\mathrm{d}\varphi+\frac{1}{12}m_2l_2^2\left(\ddot{\psi}+\ddot{\varphi}\right)\mathrm{d}\psi \end{aligned}$$

$$\sum\delta W=\left(\frac{1}{2}m_1l_1+m_2l_0\right)g\sin\varphi\mathrm{d}\varphi+M\mathrm{d}\psi$$

By the principle of work and energy in its differential form, i.e. $\mathrm{d}T=\delta W$, one gets

$$\left[\left(\frac{1}{3}m_1l_1^2+m_2l_0^2+\frac{1}{12}m_2l_2^2\right)\ddot{\varphi}+\frac{1}{12}m_2l_2^2\ddot{\psi}-\left(\frac{1}{2}m_1l_1+m_2l_0\right)g\sin\varphi\right]\mathrm{d}\varphi+\left[\frac{1}{12}m_2l_2^2\left(\ddot{\psi}+\ddot{\varphi}\right)+M\right]\mathrm{d}\psi=0$$

According to Theorem 1, letting the coefficient terms $\mathrm{d}\varphi$ and $\mathrm{d}\psi$ be zero will lead to the dynamical equations of the system.

$$\begin{cases}\left(\frac{1}{3}m_1l_1^2+m_2l_0^2+\frac{1}{12}m_2l_2^2\right)\ddot{\varphi}+\frac{1}{12}m_2l_2^2\ddot{\psi}-\left(\frac{1}{2}m_1l_1+m_2l_0\right)g\sin\varphi \\ \frac{1}{12}m_2l_2^2\left(\ddot{\psi}+\ddot{\varphi}\right)+M=0\end{cases}$$

It is easy to check that the result is the same as derived by Lagrange equation.

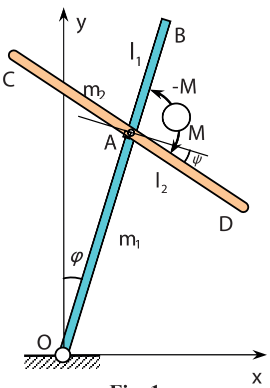

Fig. 1

3 Discussions of the failure in the absence of the restriction condition

This section provides a detailed discussion about the application of the principle of work and energy in differential form when the restrictive conditions given in the theorem are not satisfied. Two case studies are presented here to show that when the conditions are not satisfied, the application of the principle of work and energy in differential form as shown in Example 1 will generally give the wrong results, although sometimes gives right answers through the use of some tricky combination techniques.

Example 2. Homogeneous disk A of mass m_1 and radius r rolls without slipping over the ground. Bar AB of mass m_2 and length l is connected with disk A by a pin joint at point A as shown in Fig. 2. Try to build the differential equations of the system.

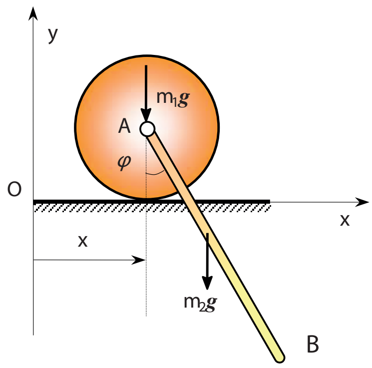

Fig. 2

Solution: Obviously, the constraints of the system are obviously bilateral, scleronomic, holonomic and ideal constraints. If the displacement of centre A of the disk and the rotating angle of the bar AB are chosen to be the generalized coordinates of this system, then the kinetic energy theorem of the system will read

$$T=\frac{1}{4}\left(3m_1+2m_2\right)\dot{x}^2+\frac{1}{2}m_2l\dot{x}\dot{\varphi}\cos\varphi+\frac{1}{6}m_2l^2\dot{\varphi}^2 \tag{9}$$

It can be seen that the generalized coordinate φ is present in the expression of T, so the condition of theorem 1 is not satisfied. However, carrying on with the same procedure as in Example 1 will lead to

$$
\begin{aligned}
\mathrm{d}T &= \frac{1}{2}\left(3m_1+2m_2\right)\dot{x}\ddot{x}\mathrm{d}t+\frac{1}{2}m_2 l\dot{\varphi}\ddot{x}\cos\varphi\mathrm{d}t \\
&+\frac{1}{2}m_2 l\dot{x}\left(\ddot{\varphi}\cos\varphi-\dot{\varphi}^2\sin\varphi\right)\mathrm{d}t+\frac{1}{3}m_2 l^2\dot{\varphi}\ddot{\varphi}\mathrm{d}t \\
&= \frac{1}{2}\left(3m_1+2m_2\right)\ddot{x}\mathrm{d}x+\frac{1}{2}m_2 l\ddot{x}\cos\varphi\mathrm{d}\varphi \\
&+\frac{1}{2}m_2 l\left(\ddot{\varphi}\cos\varphi-\dot{\varphi}^2\sin\varphi\right)\mathrm{d}x+\frac{1}{3}m_2 l^2\ddot{\varphi}\mathrm{d}\varphi \\
&= \left[\frac{1}{2}\left(3m_1+2m_2\right)\ddot{x}+\frac{1}{2}m_2 l\left(\ddot{\varphi}\cos\varphi-\dot{\varphi}^2\sin\varphi\right)\right]\mathrm{d}x \\
&+\left[\frac{1}{2}m_2 l\ddot{x}\cos\varphi+\frac{1}{3}m_2 l^2\ddot{\varphi}\right]\mathrm{d}\varphi
\end{aligned} \tag{10}
$$

and

$$
\delta W = -m_2 g\cdot\frac{l}{2}\sin\varphi\mathrm{d}\varphi
$$

Applying $\mathrm{d}T=\delta W$ and simplifying the equation, one can obtain

$$
\left[\frac{1}{2}\left(3m_1+2m_2\right)\ddot{x}+\frac{1}{2}m_2 l\left(\ddot{\varphi}\cos\varphi-\dot{\varphi}^2\sin\varphi\right)\right]\mathrm{d}x+
\left[\frac{1}{2}m_2 l\ddot{x}\cos\varphi+\frac{1}{3}m_2 l^2\ddot{\varphi}+m_2 g\cdot\frac{l}{2}\sin\varphi\mathrm{d}\varphi\right]\mathrm{d}\varphi=0
$$

Letting the coefficients of dx and dφ be zeroes gives

$$
\begin{cases}
\frac{1}{2}\left(3m_1+2m_2\right)\ddot{x}+\frac{1}{2}m_2 l\left(\ddot{\varphi}\cos\varphi-\dot{\varphi}^2\sin\varphi\right)=0 \\
\frac{1}{2}m_2 l\ddot{x}\cos\varphi+\frac{1}{3}m_2 l^2\ddot{\varphi}+m_2 g\cdot\frac{l}{2}\sin\varphi=0
\end{cases}
$$

It is easy to verify the above results with that obtained by Lagrange equation method, which shows that the above procedure could also work even when the conditions of the theorem are not satisfied. But, it is important to note that if $m_2 l\dot{x}\dot{\varphi}^2\sin\varphi\mathrm{d}t$ is simplified to $m_2 l\dot{x}\dot{\varphi}\sin\varphi\mathrm{d}\varphi$, the above procedure will lead to the following equations.

$$
\begin{cases}
3\ddot{x}\cos\varphi+2l\ddot{\varphi}+3g\sin\varphi=0 \\
\left(3m_1+2m_2\right)\ddot{x}+m_2 l\left(\ddot{\varphi}\cos\varphi-\dot{x}\dot{\varphi}\sin\varphi\right)=0
\end{cases}
$$

which is proved wrong by the Lagrange equation. Hence, it is very tricky as to how to simplify the terms involving the product of two or more generalized velocities to obtain the correct dynamical equations of a system. The improper combination of the generalized velocity with dt will lead to the wrong results.

From the above discussion, it seems that the proposed method could be applicable to some problems which do not satisfy the restrictive conditions. In what follows, an example is given to show that this method could totally fail with any combination technique when the constraint conditions are not satisfied.

Example 3. The Homogeneous bar OA of mass m_1 and length l rotates about the fixed horizontal axis Oz in the vertical plane and is acted by an external moment T_θ, as shown in Fig. 3. The collar B of mass m_2 slides along the smooth bar OA. Try to build the dynamical equations of the system.

Solution: Obviously, the constraints of the system are bilateral, scleronomic, holonomic and ideal constraints. If the rotating angle of the bar OB and distance ρ between B and O point arechosen to be generalized coordinates of this system, then the kinetic energy of the system reads

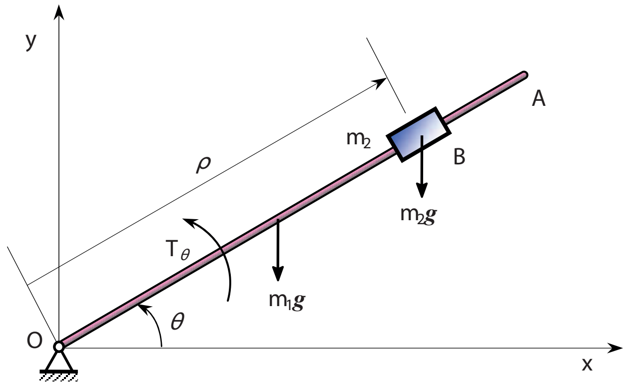

Fig. 3

$$
T=\frac{1}{2}J_0\dot{\theta}^2+\frac{1}{2}m_2\left[\dot{\rho}^2+\left(\rho\dot{\theta}\right)^2\right]=\frac{1}{6}m_1 l^2\dot{\theta}^2+\frac{1}{2}m_2\dot{\rho}^2+\frac{1}{2}m_2\rho^2\dot{\theta}^2
$$

$$
\begin{aligned}
\mathrm{d}T &= \frac{1}{3}m_1 l^2\ddot{\theta}\mathrm{d}\theta+m_2\ddot{\rho}\mathrm{d}\rho+m_2\rho\dot{\theta}^2\mathrm{d}\rho+m_2\rho^2\ddot{\theta}\mathrm{d}\theta \\
&= \left(\frac{1}{3}m_1 l^2\ddot{\theta}+m_2\rho^2\ddot{\theta}\right)\mathrm{d}\theta+\left(m_2\ddot{\rho}+m_2\rho\dot{\theta}^2\right)\mathrm{d}\rho
\end{aligned}
$$

$$
\begin{aligned}
\delta W &= T_\theta\cdot\mathrm{d}\theta-m_1 g\cdot\frac{l}{2}\cos\theta\cdot\mathrm{d}\theta \\
&-m_2 g\cdot\rho\cos\theta\cdot\mathrm{d}\theta-m_2 g\sin\theta\mathrm{d}\rho \\
&= \left(T_\theta-m_1 g\cdot\frac{l}{2}\cos\theta-m_2 g\cdot\rho\cos\theta\right)\mathrm{d}\theta-m_2 g\sin\theta\mathrm{d}\rho
\end{aligned}
$$

Applying $\mathrm{d}T=\delta W$ and simplifying the equation, one can obtain

$$
\begin{cases}
\frac{1}{3}m_1 l^2\ddot{\theta}+m_2\rho^2\ddot{\theta}+m_1 g\frac{l}{2}\cos\theta+m_2 g\cdot\rho\cos\theta-T_\theta=0 \\
m_2\ddot{\rho}+m_2\rho\dot{\theta}^2+m_2 g\sin\theta=0
\end{cases}
$$

Comparing the result with what is obtained by Lagrange equation method shows that the above equations are not correct [4]. No technology like in Example 2 seems applicable to solve the problem. Actually, because dθ and dρ are both real displacements, they are not independent in the evolution of the dynamics of the system. Hence, the linear combination of them equate zero doesn't imply the corresponding coefficients are zeros. This example also shows that the restrictive conditions in Theorem 1 are necessary when employing the principle of work and energy alone for a MDOF system.

4 Conclusions

This study discusses the applicability of Principle of Work and Energy to MDOF systems. It is well known that the principle of work and energy can be used to derive the dynamical equation of an SDOF mechanical system, but must be employed with some other principles of mechanics together to build up all the dynamical equations of a MDOF mechanical system. The conclusion of this paper refines this common sense. That is, the principle of work and energy in differential form can be employed alone to derive all the dynamical equations of a MDOF mechanical system in the case that the kinetic energy of the system does not contain the position variables of the generalized coordinates. This discovery suggests a new way other than the Lagrange equation method to build the dynamical equations for some special class of mechanical systems. It is simpler and easier to apply than Lagrange equation method in this case. However, it is important to point out that the Principle of Work and Energy only deals with the real displacement instead of the virtual displacement. The routine provided in Theorem 1 will no doubt lead to the right dynamical equations of the systems of that class, but does not mean that in the physical sense the real displacements which are represented by the differential of generalized coordinates are independent of each other in the system. That is also why this method can only apply to some special class of mechanical systems. Finally, we have to emphasize that even it is shown by an example that through the use of some tricky combination techniques, this method can also apply to the systems which do not satisfy the restrictive equation, it is strongly recommended not to use such tricks in any of such cases, at least before somebody else will give an exact proof as to the use of the tricks in general.

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