FINITE ELEMENT ANALYSIS OF ANISOTROPIC SHELLS INCLUDING LARGE DISPLACEMENTS AND LARGE ROTATIONS

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Abstract

In analysing fiber reinforced anisotropic shells it can be necessary to consider large displacements and rotations. The article describes one kind of finite element solution of the problem.

1. Introduction

Shell structures are common in practice. Geometrically nonlinear analysis can often be necessary — research on this topic is in progress, the literature is enormous [1—10].

Considering geometrical nonlinearity can be considered at different levels when analysing shell structures. In the simplest case the displacements are regarded to be finite, but the Kirchhoff hypothesis is assumed to be valid, which implies that the products of the derivatives of the displacements tangential to the shell surface are negligable. At the next level this hypothesis is not assumed, but the strains are small. At last in the most complicated case the displacements and strains are regarded to be large. Especially fiber reinforced composites produce small strains until failure, thus in the following large displacements — large rotations — small strains will be assumed.

Various researchers have adopted many different formulation strategies and procedures to accomodate large rotation capability. However, most of them are restricted to small rotations between two successive load increments during the loading process (except Surana [8]), higher load levels can only be reached by several steps. The present article gives the stiffness matrix of a degenerated shell element following Surana's method [9] by which large load increments can be prescribed.

Shell structures are often made of fiber reinforced composite materials in practice. In analysing them (in the majority of cases) the inhomogeneous material is replaced by an equivalent homogeneous one, constitutive properties of which are computed by theory. Methods of determining these constitutive constants and the related literature can be found in the excellent book by Christensen [12] and Hashins's article [13].

2. Basic equations

When employing the Total Lagrangian approach, the strain and stress fields are referred to the original geometric configuration at time $t=0$ thus the Green—Lagrange strain tensor and the II. Piola—Kirchhoff stress tensor are used. The basic equations are as follows:

$$H_{KL} = \frac{1}{2}(U_{K;L} + U_{L;K} + U_{M;K} + U_{M;L}) \tag{1}$$

$$S_{IJ} = C_{IJKL} H_{KL} \tag{2}$$

$$\int_{(V)} S_{LM} \delta H_{KL} \, dV = \delta W_e \tag{3}$$

where H_{KL}, S_{IJ} and U_I denote the Cartesian components of the Green—Lagrange strain tensor, the II. Piola—Kirchhoff stress tensor and the displacement vector respectively. C_{IJKL} are the Cartesian components of the fourth order elasticity tensor and δW_e is the virtual work of the external forces. The first is the geometric, the second is the constitutive equation and the third is the principle of virtual work.

3. Element description

The employed shell element was described for linear analysis by Ahmad et al. [1]. The element geometry is illustrated in Fig. 1. The Cartesian components of an arbitrary point inside the element are:

$$\{\mathbf{X}\} = \sum_{k=1}^{n} (\{\mathbf{X}_k\} + \zeta \frac{h_k}{2} \{\mathbf{V}_{3k}\}) N_k(\xi, \eta) = [\mathbf{N}]\{\mathbf{R}\} \tag{4}$$

where the vector $\{\mathbf{R}\}$ incorporates the coordinates of the nodal points and the unit normal vectors at the nodes:

$$\{\mathbf{R}\}^T = \{\{\mathbf{X}_1\}^T \{\mathbf{V}_{31}\}^T \ldots \{\mathbf{X}_n\}^T \{\mathbf{V}_{3n}\}^T\}. \tag{5}$$

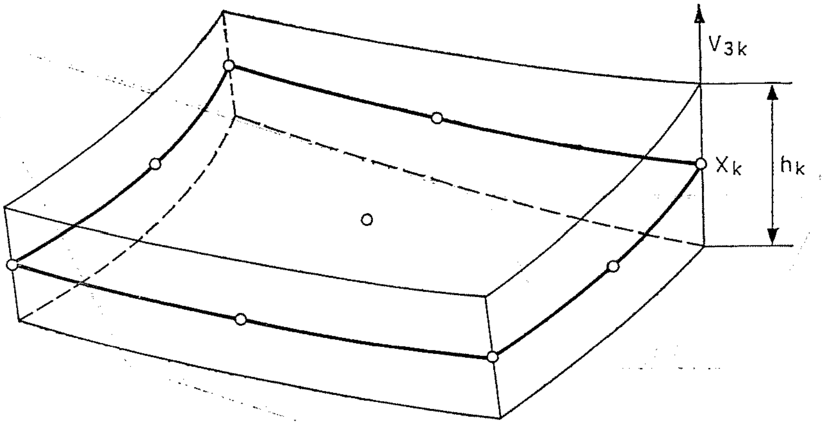

Fig. 1

Matrix $[N]$ consists of the interpolating functions, thus depends only on the ξ, η, ζ curvilinear coordinates.

Interpolating the displacement field the following assumptions are made:

— The "normals" (ζ parameter lines) remain straight during deformation

— The length of the "normals" are unchanged.

With the above assumptions:

$$\{\mathbf{u}\} = \sum_{k=1}^{n} \left(\{\overline{\mathbf{U}}_k\} + \zeta \frac{h_k}{2} \{\mathbf{p}_k\} \right) N_k(\xi, \eta) = [\mathbf{N}]\{\mathbf{U}\} \tag{6}$$

where $\{\overline{\mathbf{U}}_k\}$ is the displacement vector of the k-th node, $\{\mathbf{p}_k\}$ is the displacement of the end point of the unit vector $\{\mathbf{V}_{3k}\}$ (Fig. 2). $\{\mathbf{U}\}$ is the generalized displacement vector of the element:

$$\{\mathbf{U}\}^T = \{\{\overline{\mathbf{U}}_1\}^T \{\mathbf{p}_1\}^T \dots \{\overline{\mathbf{U}}_n\}^T \{\mathbf{p}_n\}^T\}. \tag{7}$$

In view of the assumption that the lengths of the "normals" are unchanged, the components of vector $\{\mathbf{p}\}$ are not independent, and can be computed by the help of two angles. (In the rest of this section the "k" subscript is omitted, because every vector is related to the same node of the element.) Let $\{\mathbf{V}_1\}$ and $\{\mathbf{V}_2\}$ be unit vectors, to form an orthogonal basis at the node together with $\{\mathbf{V}_3\}$. Vector $\{\mathbf{p}\}$ can be evaluated from the angles α and β (Fig. 3). Let

$$\Phi_1 \equiv \alpha - \frac{\pi}{2} \quad \Phi_2 \equiv \beta. \tag{8}$$

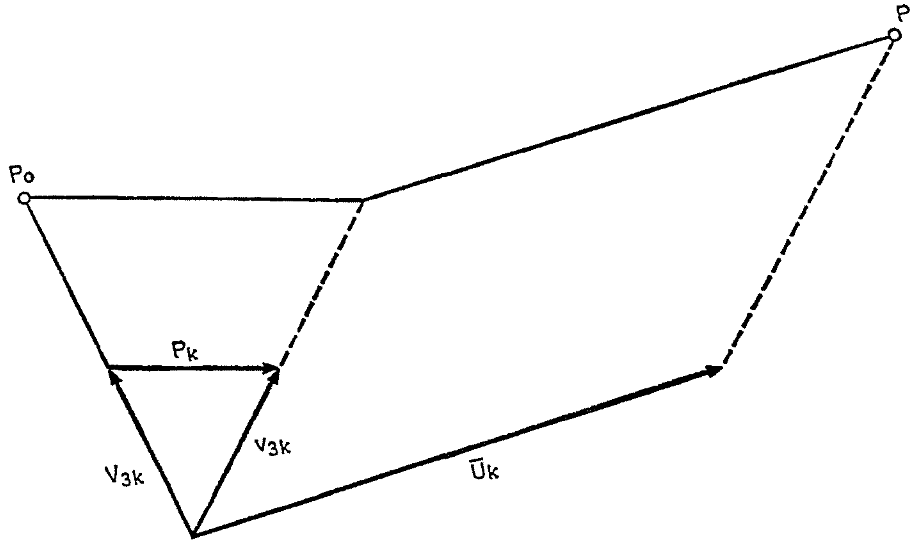

Fig. 2

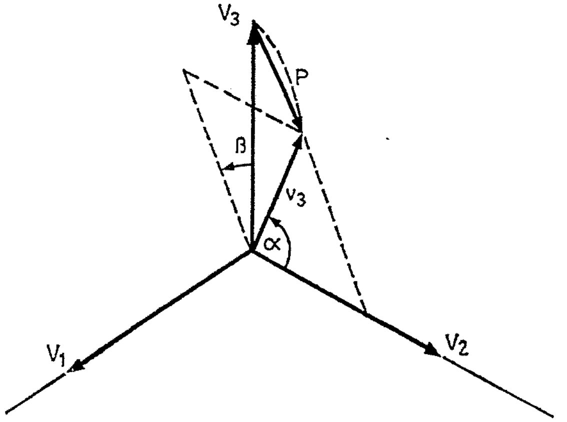

Fig. 3

Thus vector {p} is the following (in the $\{V_1\}$, $\{V_2\}$, $\{V_3\}$ basis).

$$\{\mathbf{p}\} = \begin{bmatrix} \cos \Phi_1 \sin \Phi_2 \\ -\sin \Phi_1 \\ \cos \Phi_1 \cos \Phi_2 - 1 \end{bmatrix} \tag{9}$$

In the general case the displacements are nonlinear functions of the rotations. (In the classical small displacement — small strain case $\Phi_1 \ll 1$, $\Phi_2 \ll 1$, thus $\sin \Phi_1 \approx \Phi_1$, $\sin \Phi_2 \approx \Phi_2$, and $\cos \Phi_1 \approx 1$, $\cos \Phi_2 \approx 1$, the displacements are linearly related to the rotations.)

4. Strains and stresses

The Cartesian components of the Green—Lagrange strain tensor and the II Piola—Kirchhoff stress tensor can be put into vector form:

$$\begin{aligned} \{\varepsilon\}^T &= [H_{11} H_{22} H_{33} H_{12} H_{23} H_{31}] = [\varepsilon_1 \dots \varepsilon_6] \\ \{\sigma\}^T &= [S_{11} S_{22} S_{33} S_{12} S_{23} S_{31}] = [\sigma_1 \dots \sigma_6]. \end{aligned} \tag{10}$$

The two above vectors are connected by the constitutive matrix, which incorporates the elements of the C tensor:

$$\{\sigma\} = [\mathbf{D}]\{\varepsilon\}. \tag{11}$$

In calculating the stresses in shells, it is usual to neglect the σ stresses normal to the shell midsurface. Let us consider the (11) relation at an arbitrary point of the element in a local $x'y'z'$ coordinate system, where the direction of the z' axis is parallel to the normal. Thus

$$\begin{aligned} \sigma_{z'} &\approx 0 \\ \{\sigma'\} &= [\mathbf{D}']\{\varepsilon'\} \\ [\mathbf{D}] &= [\mathbf{T}][\mathbf{D}'][\mathbf{T}]^T. \end{aligned} \tag{12}$$

Elements of the $[T]$ matrix can be determined by means of the direction cosines between the axes of the coordinate systems $x'y'z'$ and xyz.

The strain vector can be evaluated from the displacement gradient. Let

$$\begin{aligned} \{\mathbf{g}\}^T &= \{\{\mathbf{g}_x\}^T \{\mathbf{g}_y\}^T \{\mathbf{g}_z\}^T\} = [g_1 \dots g_9] \\ \{\mathbf{g}_x\} &= \frac{\partial\{\mathbf{u}\}}{\partial x} \quad \{\mathbf{g}_y\} = \frac{\partial\{\mathbf{u}\}}{\partial y} \quad \{\mathbf{g}_z\} = \frac{\partial\{\mathbf{u}\}}{\partial z}. \end{aligned} \tag{13}$$

Thus

$$\{\varepsilon\} = [\mathbf{H}]\{\mathbf{g}\} + \frac{1}{2}[\mathbf{M}(\{\mathbf{g}\})]\{\mathbf{g}\} \tag{14}$$

where

$$[H] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[M] = \begin{bmatrix} g_1 & g_2 & g_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_4 & g_5 & g_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_7 & g_8 & g_9 \\ g_4 & g_5 & g_6 & g_1 & g_2 & g_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_7 & g_8 & g_9 & g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 & 0 & 0 & 0 & g_1 & g_2 & g_3 \end{bmatrix} \quad (15)$$

$$\{\mathrm{g}\} = [\mathrm{S}]\{\mathrm{U}\}$$

$$[S] = \begin{bmatrix} \dfrac{\partial [\mathrm{N}]}{\partial x} \\ \dfrac{\partial [\mathrm{N}]}{\partial y} \\ \dfrac{\partial [\mathrm{N}]}{\partial z} \end{bmatrix}.$$

Matrix $[S]$ can be calculated from the derivatives of the interpolating functions and the inverse of the Jacobian.

5. Variations

The variation of strains are as follows

$$\delta\{\varepsilon\} = \frac{\partial\{\varepsilon\}}{\partial\{\mathrm{g}\}}\delta\{\mathrm{g}\} = ([\mathrm{H}]+[\mathrm{M}])\,\delta\{\mathrm{g}\}$$

$$\delta\{\mathrm{g}\} = \frac{\partial\{\mathrm{g}\}}{\partial\{\mathrm{U}\}}\delta\{\mathrm{U}\} = [\mathrm{S}]\,\delta\{\mathrm{U}\} \quad (16)$$

$$\delta\{\mathrm{U}\} = \frac{\partial\{\mathrm{U}\}}{\partial\{\mathrm{a}\}}\delta\{\mathrm{a}\} = [\mathrm{Q}]\,\delta\{\mathrm{a}\}$$

where $\{\mathrm{a}\}$ is the vector of the nodal unknowns:

$$\{\mathrm{a}\}^T = [\{\mathrm{a}_1\}^T \dots \{\mathrm{a}_n\}^T]$$

$$\{\mathrm{a}_k\}^T = [U_k V_k W_k \Phi_{1k} \Phi_{2k}] \quad (17)$$

and [Q] is

$$[\mathbf{Q}] = \mathrm{diag}\,([\mathbf{Q}_1] \dots [\mathbf{Q}_n])$$

$$[\mathbf{Q}_k] = \frac{\partial\{\mathbf{U}_k\}}{\partial\{\mathbf{a}_k\}} = \begin{bmatrix} [1] & [0] \\ [0] & [\mathbf{V}_k] \end{bmatrix} \tag{18}$$

$$[\mathbf{V}_k] = \left[\frac{\partial\{\mathbf{p}_k\}}{\partial\Phi_{1k}} \; \frac{\partial\{\mathbf{p}_k\}}{\partial\Phi_{2k}}\right].$$

Matrix $[B]$ depends not only on the ξ, η, ζ curvilinear coordinates (as matrices $[N]$ and $[S]$), but on the nodal unknowns, too. According to (16)—(18):

$$\delta\{\varepsilon\} = ([\mathbf{H}]+[\mathbf{M}])\,[\mathbf{S}]\,[\mathbf{Q}]\,\delta\{\mathbf{a}\} = ([\mathbf{B}^0]+[\mathbf{B}^{nl}])\,\delta\{\mathbf{a}\} = [\mathbf{B}]\,\delta\{\mathbf{a}\} \tag{19}$$

6. Solution of the equations

Summing up, the discretised basic equations are the following:

$$\{\varepsilon\} = \left([\mathbf{H}]+\frac{1}{2}[\mathbf{M}]\right)[\mathbf{S}]\,\{\mathbf{U}\} \tag{20}$$

$$\{\sigma\} = [\mathbf{D}]\,\{\varepsilon\} \tag{21}$$

$$\{\mathbf{P}\} = \int_{(V)} [\mathbf{B}]^T\{\sigma\}\,dV = \{\mathbf{F}\}. \tag{22}$$

Equations (20)—(22) are nonlinear, thus incremental — iterative solution strategy can be used. The incremental form of equation (22) is:

$$d\{\Psi\} = d\{\mathbf{P}\}-d\{\mathbf{F}\} = \int_{(V)} [\mathbf{B}]^T d\{\sigma\}+d[\mathbf{B}]^T\{\sigma\}\,dV-d\{\mathbf{F}\} \tag{23}$$

where $d\{\Psi\}$ is the incremental residual force. Let us consider the sum in the integral on the right side of (23).

$$[\mathbf{B}]^T d\{\sigma\} = [\mathbf{B}]^T\frac{\partial\{\sigma\}}{\partial\{\varepsilon\}}\frac{\partial\{\varepsilon\}}{\partial\{\mathbf{a}\}}\,d\{\mathbf{a}\} = [\mathbf{B}]^T[\mathbf{D}]\,[\mathbf{B}]\,d\{\mathbf{a}\}. \tag{24}$$

Thus

$$\int_{(V)} [\mathbf{B}]^T d\{\sigma\}\,dV = [\mathbf{K}^0]\,d\{\mathbf{a}\} \tag{25}$$

where

$$[\mathbf{K}^0] = \int_{(V)} [\mathbf{B}]^T[\mathbf{D}]\,[\mathbf{B}]\,dV. \tag{26}$$

The second part of the integral in (23) is:

$$\begin{gathered} d[\mathbf{B}]^T\{\sigma\} = d(([\mathbf{H}]+[\mathbf{M}])\,[\mathbf{S}]\,[\mathbf{Q}])^T\{\sigma\} = \\ [\mathbf{Q}]^T[\mathbf{S}]^T d[\mathbf{M}]^T\{\sigma\}+d[\mathbf{Q}]^T[\mathbf{S}]^T([\mathbf{H}]^T+[\mathbf{M}]^T)\{\sigma\}, \end{gathered} \tag{27}$$

Considering the first part of the sum on the right side of (27), let

$$\{\mathbf{q}\} = [\mathbf{M}]^T\{\boldsymbol{\sigma}\}. \tag{28}$$

Thus

$$d[\mathbf{M}]^T\{\boldsymbol{\sigma}\} = \frac{\partial\{\mathbf{q}\}}{\partial\{\mathbf{g}\}}\frac{\partial\{\mathbf{g}\}}{\partial\{\mathbf{U}\}}\frac{\partial\{\mathbf{U}\}}{\partial\{\mathbf{a}\}}\,d\{\mathbf{a}\} = [\boldsymbol{\sigma}][\mathbf{S}][\mathbf{Q}]\,d\{\mathbf{a}\} \tag{29}$$

where

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_1[\mathbf{1}] & \sigma_4[\mathbf{1}] & \sigma_6[\mathbf{1}] \\ \sigma_4[\mathbf{1}] & \sigma_2[\mathbf{1}] & \sigma_5[\mathbf{1}] \\ \sigma_6[\mathbf{1}] & \sigma_5[\mathbf{1}] & \sigma_3[\mathbf{1}] \end{bmatrix}. \tag{30}$$

Thus

$$\int\limits_{(V)}[\mathbf{Q}]^T[\mathbf{S}]^T d[\mathbf{M}]^T\{\boldsymbol{\sigma}\}\,dV = [\mathbf{K}_{g1}]\,d\{\mathbf{a}\} \tag{31}$$

$$[\mathbf{K}_{g1}] = \int\limits_{(V)}[\mathbf{Q}]^T[\mathbf{S}]^T[\boldsymbol{\sigma}][\mathbf{S}][\mathbf{Q}]\,dV. \tag{32}$$

Matrix $[K_{g1}]$ is the first part of the geometric stiffness matrix, and is symmetric. Considering the second part of (26) let

$$\begin{aligned} \{\mathbf{w}\} &= [\mathbf{S}]^T([\mathbf{H}]+\mathbf{M}])\{\boldsymbol{\sigma}\} \\ \{\mathbf{z}\} &= [\mathbf{Q}]^T\{\mathbf{w}\} \end{aligned} \tag{33}$$

thus

$$d[\mathbf{Q}]^T\{\mathbf{w}\} = \frac{\partial\{\mathbf{z}\}}{\partial\{\mathbf{a}\}}\,d\{\mathbf{a}\}. \tag{34}$$

As

$$\begin{aligned} [\mathbf{Q}] &= \operatorname{diag}([\mathbf{Q}_1], \ldots, [\mathbf{Q}_n]) \\ \{\mathbf{w}\}^T &= \{\{\mathbf{w}_1\}^T \ldots \{\mathbf{w}_n\}^T\} \end{aligned} \tag{35}$$

thus

$$\{\mathbf{z}\}^T = \{\{\mathbf{z}_1\}^T \ldots \{\mathbf{z}_n\}^T\} \tag{36}$$

where

$$\{\mathbf{z}_k\} = [\mathbf{Q}_k]^T\{\mathbf{w}_k\} = \begin{bmatrix} [\mathbf{1}] & [0] \\ [0] & [\mathbf{V}_k]^T \end{bmatrix}\begin{bmatrix} \{\mathbf{w}_k'\} \\ \{\mathbf{w}_k''\} \end{bmatrix}. \tag{37}$$

Thus

$$\frac{\partial\{\mathbf{z}_k\}}{\partial\{\mathbf{a}_k\}} = [\mathbf{P}_k]\{\mathbf{w}_k\} = \begin{bmatrix} [0] & [0] \\ [0] & [\tilde{\mathbf{P}}_k] \end{bmatrix}\begin{bmatrix} \{\mathbf{w}_k'\} \\ \{\mathbf{w}_k''\} \end{bmatrix} \tag{38}$$

where

$$[\tilde{\mathbf{P}}_k] = \begin{bmatrix} \dfrac{\partial^2\{\mathbf{p}_k\}^T}{\partial\Phi_1^2}\{\mathbf{w}_k''\} & \dfrac{\partial^2\{\mathbf{p}_k\}^T}{\partial\Phi_1\partial\Phi_2}\{\mathbf{w}_k''\} \\ \dfrac{\partial^2\{\mathbf{p}_k\}^T}{\partial\Phi_1\partial\Phi_2}\{\mathbf{w}_k''\} & \dfrac{\partial^2\{\mathbf{p}_k\}^T}{\partial\Phi_2^2}\{\mathbf{w}_k''\} \end{bmatrix} \tag{39}$$

thus

$$\int\limits_{(V)} d[\mathbf{Q}]^T[\mathbf{S}]^T([\mathbf{H}]^T+[\mathbf{M}]^T)\{\boldsymbol{\sigma}\}\,dV = [\mathbf{K}_{g2}]\,d\{\mathbf{a}\} \tag{40}$$

where matrix $[K_{g2}]$ is symmetric, and differs from zero only in the rotation part of the diagonal submatrices

$$[\mathbf{K}_{g2}] = \int_{(V)} \operatorname{diag}([\mathbf{P}_1], \ldots, [\mathbf{P}_n])\, dV. \tag{41}$$

Thus

$$d\{\Psi\} = ([\mathbf{K}^0]+[\mathbf{K}_{g1}]+[\mathbf{K}_{g2}])\, d\{\mathbf{a}\} = [\mathbf{K}_t]\, d\{\mathbf{a}\}. \tag{42}$$

In a lot of cases the matrix $[K_{g2}]$ is not considered. Neglecting it, the matrix $[K_t]$ will not be exactly tangent, and the solution procedure can diverge if large load steps are used.

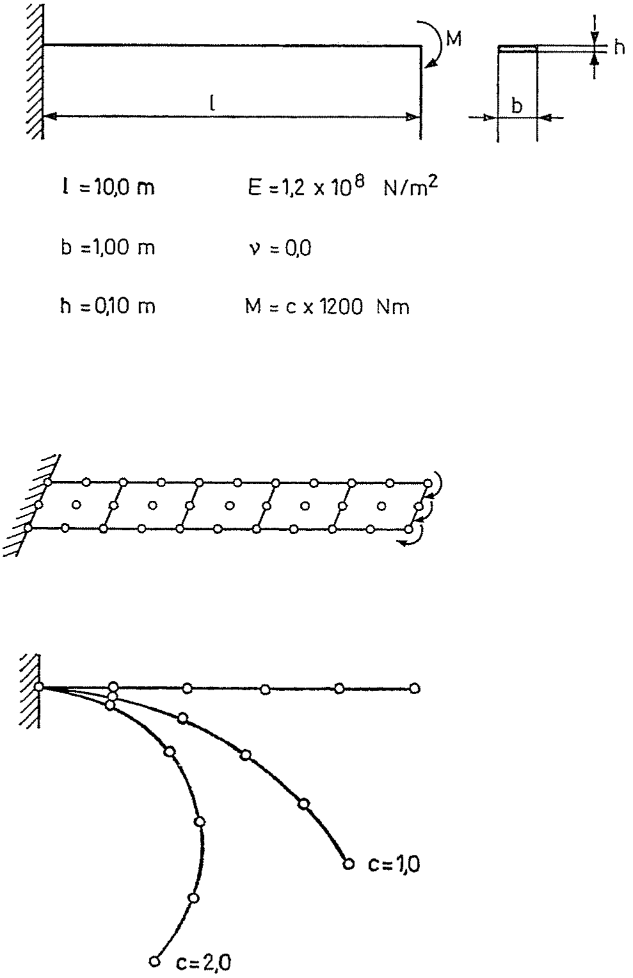

Fig. 4

7. Example

The versatility of the method described in the above sections can be demonstrated by the following simple numerical example. Let us consider a straight canteliver, loaded with the end moment M (Fig. 4). The deformed shape of the canteliver is a circular arc according to the analytical solution.

The algorithm is implemented in the SHEILA finite element code, running on IBM PC—AT. In the finite element model five elements were used. The deformed shape which can be seen in Fig. 4 could be reached in two load steps. Very large displacements and rotations occurred during the loading process.

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