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A PRACTICAL APPROACH TO THE AFFINE TRANSFORMATIONS OF THE EUCLIDEAN PLANE[1]

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Dedicated to Professor Julius Strommer on the occasion of his 75^{th} birthday

Abstract

The aim of this paper is to give an elementary treatment of a classical item which plays central role in the applied geometry. The actual need for a more or less new presentation of such a well-known subject is explained by the fact that the use of computers easily allows us to work with any given affine transformation in a suitably (canonically) chosen coordinate-system. The choice of that new orthonormal coordinate-system is based on the diagonalization process of the Gram matrix belonging to the linear part of the transformation, and on the change of the origin for an eventual fixpoint of the given affine transformation. Nevertheless, the entire classification of the considered transformations could be given here by elementary algebraic tools.

To simplify our discussion we omit too technical denotations and details, and restrict ourselves to plane geometry. For other aspects we refer e.g. to [4] in this volume.

Keywords: affine transformation, linear algebra, Gram matrix.

Introduction

It would be hardly possible to make a complete list of all the authors who have contributed to the topic since the last century, we only mention here some of the corresponding works published in Hungary [1],[2],[3].

In order to give an affine transformation

$$\mathcal{A} : E^2 \to \mathcal{A}(E^2) \subseteq E^2$$

of the Euclidean plane E^2 it is enough to give the image of a non-degenerated triangle $P_0P_1P_2$ by three corresponding points

$$P_0' = \mathcal{A}(P_0)\ , \quad P_1' = \mathcal{A}(P_1)\ , \quad P_2' = \mathcal{A}(P_2)\ .$$

Then for any $P \in E^2$ one should have the equality

$$\overrightarrow{P_0'P'} = x\,\overrightarrow{P_0'P_1'} + y\overrightarrow{P_0'P_2'}\ ,$$

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where $P' = \mathcal{A}(P)$ and $x, y \in \mathbf{R}$ are the unique coefficients in the decomposition

$$\overrightarrow{P_0P} = x\,\overrightarrow{P_0P_1} + y\,\overrightarrow{P_0P_2}\ .$$

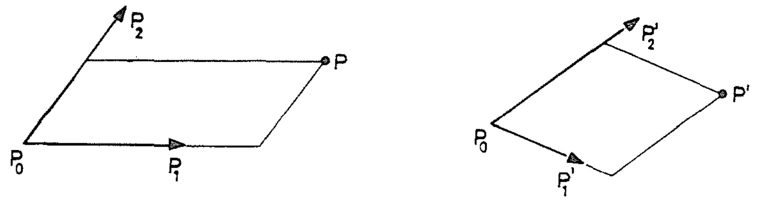

Fig. 1.

Especially, if the affine transformation is given with respect to a fixed orthonormal coordinate-system $\{O, \mathbf{e}_1, \mathbf{e}_2\}$ we have the following equations

$$x' = a_{11}x + a_{12}y + x_0,$$
$$y' = a_{21}x + a_{22}y + y_0,$$

or in matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} ,$$

where $P'(x'; y')$ is the image of the point $P(x; y)$.

It can be noticed that the given affine transformation $\mathcal{A}$ is still uniquely characterized by the fact that the coordinates of the point P with respect to the non-degenerated triangle

$$O(0;0)\ , \quad E_1(1;0)\ , \quad E_2(0;1)$$

are exactly the same as the coordinates of the point P' with respect to the triangle

$$O'(x_0; y_0)\ , \quad E_1'(a_{11} + x_0; a_{21} + y_0)\ , \quad E_2'(a_{12} + x_0; a_{22} + y_0).$$

We should notice here, however, that the points O', E_1', E_2' are allowed to be collinear, as well. (This is the case when $\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21} = 0$.)

So, the above defined transformation is still called an affine transformation in this paper if it is degenerated (i.e. if it maps the plane E^2 onto a straight line or onto a single point.)

Canonical Orthonormal Base

In this chapter we are looking for the smallest possible value of the angle ϑ $(-\frac{\pi}{2} < \vartheta \leq \frac{\pi}{2})$ by which the originally given orthonormal base $(\mathbf{e}_1, \mathbf{e}_2)$ should be turned to get a new (canonically adapted) orthonormal base

$$\mathbf{s}_1 = \cos\vartheta \mathbf{e}_1 + \sin\vartheta \mathbf{e}_2,$$
$$\mathbf{s}_2 = -\sin\vartheta \mathbf{e}_1 + \cos\vartheta \mathbf{e}_2$$

having the following properties:

The vectors $\mathbf{s}_1'$ and $\mathbf{s}_2'$ – images of $\mathbf{s}_1$ and $\mathbf{s}_2$ by the given transformation – are orthogonal and $|\mathbf{s}_1'| \geq |\mathbf{s}_2'|$ holds.

Let us introduce first some usual notations:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \varepsilon = \operatorname{sgn}\,(a_{11}a_{22} - a_{12}a_{21}),$$

$$\mathbf{e}_1' = \overrightarrow{O'E_1'} \quad \text{with} \quad a_1 = |\mathbf{e}_1'| = \sqrt{a_{11}^2 + a_{21}^2}\ ,$$
$$\mathbf{e}_2' = \overrightarrow{O'E_2'} \quad \text{with} \quad a_2 = |\mathbf{e}_2'| = \sqrt{a_{12}^2 + a_{22}^2},$$

and

$$\cos\varphi = \frac{\langle \mathbf{e}_1', \mathbf{e}_2' \rangle}{a_1\, a_2} \quad (0 \leq \varphi \leq \pi). \qquad \text{See } \textit{Fig. } 2$$

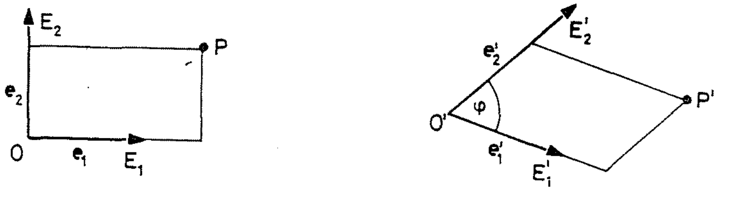

Fig. 2.

For our purpose mentioned above we have to diagonalize the symmetric Gram matrix given by

$$\mathbf{G} = \mathbf{A}^T \mathbf{A} = \begin{pmatrix} \langle \mathbf{e}_1', \mathbf{e}_1' \rangle & \langle \mathbf{e}_1', \mathbf{e}_2' \rangle \\ \langle \mathbf{e}_2', \mathbf{e}_1' \rangle & \langle \mathbf{e}_2', \mathbf{e}_2' \rangle \end{pmatrix},$$

where T denotes transposition.

From the characteristic equation

$$\lambda^2 - (a_1^2 + a_2^2)\lambda + a_1^2 a_2^2 \sin^2\varphi = 0$$

we have the following eigen-values

$$\lambda_1 = \frac{a_1^2 + a_2^2 + |D|}{2} \quad \text{and} \quad \lambda_2 = \frac{a_1^2 + a_2^2 - |D|}{2},$$

$$(\lambda_1 \geq \lambda_2 \geq 0),$$

where

$$D^2 = (a_1^2 + a_2^2)^2 - 4a_1^2 a_2^2 \sin^2\varphi \quad (0 \leq \varphi \leq \pi)\,.$$

The corresponding eigen-vectors are given by the column matrices

$$\mathbf{s}_1 = (\cos\vartheta, \sin\vartheta)^T \quad \text{and} \quad \mathbf{s}_2 = (-\sin\vartheta, \cos\vartheta)^T$$

$$\left(-\frac{\pi}{2} < \vartheta \leq \frac{\pi}{2}\right)\,.$$

In the special case when $\langle \mathbf{e}_1', \mathbf{e}_2' \rangle = 0$ (i.e. $\cos\varphi = 0$)

$$\lambda_1 = \max(a_1^2, a_2^2) \quad \text{and} \quad \lambda_2 = \min(a_1^2, a_2^2)$$

hold, and the corresponding eigen-vectors are given by

$$\vartheta = 0 \quad \text{if} \quad a_1 \geq a_2 \quad \text{and} \quad \vartheta = \frac{\pi}{2} \quad \text{if} \quad a_1 < a_2\,.$$

In the other case when $\langle \mathbf{e}_1', \mathbf{e}_2' \rangle \neq 0$ (i.e. $\cos\varphi \neq 0$) the corresponding eigen-directions are given by

$$\operatorname{tg}\vartheta = \frac{2a_1 a_2 \cos\varphi}{a_1^2 - a_2^2 + |D|} = \frac{a_2^2 - a_1^2 + |D|}{2a_1 a_2 \cos\varphi}$$

and

$$\operatorname{tg}\left(\vartheta + \frac{\pi}{2}\right) = \frac{2a_1 a_2 \cos\varphi}{a_1^2 - a_2^2 - |D|} = \frac{a_2^2 - a_1^2 - |D|}{2a_1 a_2 \cos\varphi}.$$

It is easy to see that sgn (tg ϑ) = sgn ($\cos\varphi$). Especially, in the case of a degenerated transformation

$$\operatorname{tg}\vartheta = \frac{a_2}{a_1} \quad \text{if} \quad \varphi = 0\,,$$

$$\text{and}\,\operatorname{tg}\vartheta = -\frac{a_1}{a_2} \quad \text{if} \quad \varphi = \pi$$

hold. In the final special case when $a_1 = a_2 \neq 0$ ($\cos\varphi \neq 0$) we have

$$\vartheta = \frac{\pi}{4} \quad \text{if} \quad 0 \le \varphi < \frac{\pi}{2} \quad \text{and} \quad \vartheta = -\frac{\pi}{4} \quad \text{if} \quad \frac{\pi}{2} < \varphi \le \pi \,.$$

So in any case we could introduce the eigen-vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ of the Gram matrix $\mathbf{G}$ corresponding to the eigen-values λ_1 and λ_2 ($\lambda_1 \ge \lambda_2 \ge 0$). It is easy to see the validity of the following relations:

$$\begin{aligned} \langle \mathbf{s}_1, G(\mathbf{s}_1)\rangle &= \langle A(\mathbf{s}_1), A(\mathbf{s}_1)\rangle = \lambda_1 \,, \\ \langle \mathbf{s}_2, G(\mathbf{s}_2)\rangle &= \langle A(\mathbf{s}_2), A(\mathbf{s}_2)\rangle = \lambda_2, \\ \text{and} \quad \langle \mathbf{s}_1, G(\mathbf{s}_2)\rangle &= \langle A(\mathbf{s}_1), A(\mathbf{s}_2)\rangle = 0 \,, \end{aligned}$$

where G and A denote the linear transformations whose matrices in the given orthonormal base $(\mathbf{e}_1, \mathbf{e}_2)$ are $\mathbf{G}$ and $\mathbf{A}$, respectively.

Consequently, the linear part of the given affine transformation can be canonically expressed by the following equations:

$$\begin{aligned} \mathbf{s}_1' &= \mu_1 \cos\alpha \mathbf{s}_1 + \mu_1 \sin\alpha \mathbf{s}_2 \\ \text{and} \mathbf{s}_2' &= -\varepsilon\mu_2 \sin\alpha \mathbf{s}_1 + \varepsilon\mu_2 \cos\alpha \mathbf{s}_2, \end{aligned}$$

where

$$\mu_1 = \sqrt{\lambda_1} = |\mathbf{s}_1'| \,, \quad \mu_2 = \sqrt{\lambda_2} = |\mathbf{s}_2'| \,,$$

$\varepsilon = \operatorname{sgn}(a_{11}a_{22} - a_{12}a_{21})$ and the angle α ($-\pi < \alpha \le \pi$) is explicitly given by the relations

$$\langle \mathbf{s}_1, \mathbf{s}_1'\rangle = \mu_1 \cos\alpha \quad \text{and} \quad \langle \mathbf{s}_2, \mathbf{s}_1'\rangle = \mu_1 \sin\alpha \,. \quad \text{See} \quad \textit{Fig. 3}$$

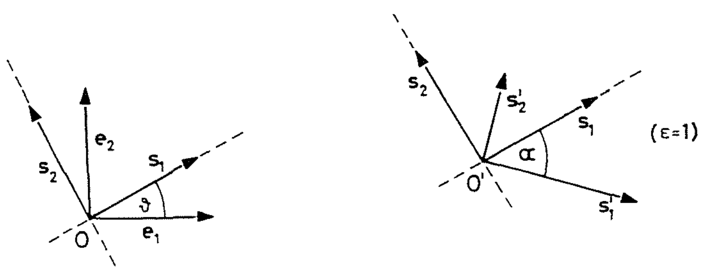

Fig. 3.

It is well-known that the transformation preserves or reverses the orientation of the plane E^2 according to the cases $\varepsilon = 1$ or $\varepsilon = -1$, respectively. In the case of a degenerated transformation when $a_{11}a_{22} = a_{12}a_{21}$ the vector $\mathbf{s}_2'$ vanishes, or both $\mathbf{s}_2'$ and $\mathbf{s}_1'$ vanish.

We should mention here also the fact that in the special case when $\lambda_1 = \lambda_2$ (i.e. $\mu_1 = \mu_2$) the choice of a canonical orthonormal base $(\mathbf{s}_1, \mathbf{s}_2)$ is arbitrary. Our choice has been fixed here by $\vartheta = 0$ (i.e. the originally given base $(\mathbf{e}_1, \mathbf{e}_2)$ is left unchanged.) It is easy to see that the angle α is independent of our choice of the canonical orthonormal base since the relation $\lambda_1 = \lambda_2$ implies

$$a_1 = a_2 \quad \text{and} \quad \cos\varphi = 0\ .$$

Until this point we could therefore uniquely determine the geometric parameters $\vartheta, \alpha, \mu_1, \mu_2$ (together with ε) of the linear part of the transformation $\mathcal{A}$ by the help of the entries $a_{11}, a_{12}, a_{21}, a_{22}$ of the matrix $\mathbf{A}$. (See *Fig. 3*).

Ordinary Affine Transformations

After having investigated the linear part of the given affine transformation $\mathcal{A}$, we are looking now for a point $F \in E^2$ such that

$$\mathcal{A}(F) = F' = F$$

is valid. So we are looking for the solutions (ξ, η) of the following system of linear equations:

$$\xi\mathbf{s}_1 + \eta\mathbf{s}_2 = \overrightarrow{OO'} + \xi\mathbf{s}_1' + \eta\mathbf{s}_2'.$$

It is easy to see that we have a unique solution for the fixpoint F if and only if the vectors $\mathbf{v}_1 = \mathbf{s}_1' - \mathbf{s}_1$ and $\mathbf{v}_2 = \mathbf{s}_2' - \mathbf{s}_2$ are linearly independent. (See *Figs. 4* and *5*).

Consequently it is reasonable to distinguish the ordinary linear transformations, where the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent from the special ones where the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. In the same way the affine transformation $\mathcal{A}$ is called *ordinary* if there is exactly one fixpoint F, and *special* if there is no fixpoint or the number of fixpoints is infinite. (In this latter case the vector $\overrightarrow{OO'}$ should be in the subspace spanned by the linearly dependent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.)

It is evident that an ordinary affine transformation can always be considered simply as a linear transformation. In its canonical coordinate system $\{F, \mathbf{s}_1, \mathbf{s}_2\}$ the matrix $\mathbf{S}$ of the transformation is given by

$$\mathbf{S} = \begin{pmatrix} \mu_1 \cos\alpha & -\varepsilon\mu_2 \sin\alpha \\ \mu_1 \sin\alpha & \varepsilon\mu_2 \cos\alpha \end{pmatrix}$$

with the following condition:

$$1 + \varepsilon\mu_1\mu_2 \neq (\mu_1 + \varepsilon\mu_2)\cos\alpha \ ,$$

$$\text{(or in other terms det } (\mathbf{S} - \mathbf{Id}) \neq 0.)$$

It should be noticed that the fact whether det $(\mathbf{A} - \mathbf{Id})$ vanishes or not does not depend on the choice of the base.

In order to classify the ordinary affine transformations we have only a few cases to distinguish:

1. Degenerated transformations
   - 1.1. $\mu_1 = \mu_2 = 0 \Rightarrow \mathcal{A}(E^2)$ is the single point F.
   - 1.2. $\mu_1 > \mu_2 = 0$ and $\mu_1 \cos\alpha \neq 1 \Rightarrow \mathcal{A}(E^2)$ is a straight line passing through the point F.
2. Non-degenerated transformations
   - 2.1. $\mu = \mu_1 = \mu_2 > 0$, $\varepsilon = 1$ and $1 + \mu^2 \neq 2\mu\cos\alpha$.
     We have in this class all the orientation *preserving* similarities. It is easy to see that the isometries are included except the identity since $1 + \mu^2 = 2\mu\cos\alpha$ holds only for $\mu = 1$ and $\alpha = 0$.
   - 2.2. $\mu = \mu_1 = \mu_2 > 0$, $\varepsilon = -1$ and $1 - \mu^2 \neq 0$.
     We have in this class all the orientation *reversing* similarities except the isometries since $\mu \neq 1$.
   - 2.3. $\mu_1 > \mu_2 > 0$ and $1 + \varepsilon\mu_1\mu_2 \neq (\mu_1 + \varepsilon\mu_2)\cos\alpha$.
     We have in this class the general ordinary affine transformations. The orientation is preserved for $\varepsilon = 1$ and changed for $\varepsilon = -1$.

It is evident that any ordinary affine transformation can be uniquely described in its canonical coordinate system by three geometric parameters, namely by the axial dilatations μ_1, $\varepsilon\mu_2$ and by the angle α of a rotation around the fixpoint F.

Special Affine Transformations

Our aim in this last chapter is to give a classification of the special affine transformations. First we consider only the linear part of any given special

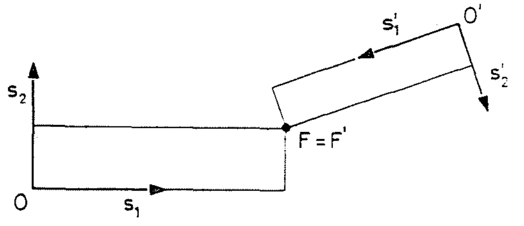

Fig. 4.

affine transformation (i.e. the vector $\overrightarrow{OO}'$ is supposed to be zero). Then this linear transformation is given in its canonical base by the following matrix:

$$\mathbf{S} = \begin{pmatrix} \mu_1 \cos\alpha & -\varepsilon\mu_2 \sin\alpha \\ \mu_1 \sin\alpha & \varepsilon\mu_2 \cos\alpha \end{pmatrix},$$

where

$$1 + \varepsilon\mu_1\mu_2 = (\mu_1 + \varepsilon\mu_2)\cos\alpha \quad \text{holds .}$$

For the classification we will distinguish the following possibilities: ($\mu_1 \geq \mu_2 \geq 0$, $\varepsilon = \pm 1$ and $-1 \leq \cos\alpha \leq 1$)

1. $\mu_1 + \varepsilon\mu_2 = 0 \implies \mu_1 = \mu_2 = 1\,, \quad \varepsilon = -1$ and the angle α is arbitrary.

2. $|\cos\alpha| = 1 \;\; (\mu_1 + \varepsilon\mu_2 > 0) \Rightarrow (\mu_1 - 1)(\varepsilon\mu_2 - 1) = 0$ or $(\mu_1 + 1)(\varepsilon\mu_2 + 1) = 0$ holds.

3. $\cos\alpha = 0 \;\; (\mu_1 + \varepsilon\mu_2 > 0) \implies \mu_1\,\mu_2 = 1$ and $\varepsilon = -1$ hold .

4. $0 < |\cos\alpha| < 1 \;\; (\mu_1 + \varepsilon\mu_2 > 0) \implies 1 + \varepsilon\mu_1\mu_2 > 0$ and $(\mu_1 - 1)(\varepsilon\mu_2 - 1) < 0$ or $1 + \varepsilon\mu_1\mu_2 < 0$ and $(\mu_1 + 1)(\varepsilon\mu_2 + 1) > 0$ hold.

In the above classification of the special linear transformations the identity, which is characterized by $\mu_1 = \mu_2 = 1$, $\varepsilon = 1$ and $\cos\alpha = 1$, can be found in class 2. Evidently, the set of fixpoints for the identity is the whole plane E^2.

In any other cases the (linearly dependent) vectors $\mathbf{v}_1 = \mathbf{s}_1' - \mathbf{s}_1$ and $\mathbf{v}_2 = \mathbf{s}_2' - \mathbf{s}_2$ uniquely define a direction called the *direction of the affinity.* Using the coordinate system $\{O, \mathbf{s}_1, \mathbf{s}_2\}$ the fixpoints of the considered linear transformation are found as solutions of the equation (see *Fig. 4* or *Fig. 5*):

$$\xi \mathbf{v}_1 + \eta \mathbf{v}_2 = \mathbf{0}\ .$$

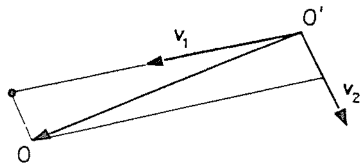

Fig. 5.

So the set of the fixpoints $F(\xi; \eta)$ is a straight line called the *axis of the affinity.* The equation of the axis is then simply given by

$$\begin{aligned} &\xi = 0 && \text{if} \quad \mathbf{v}_2 = \mathbf{0} \quad (\text{and} \quad \mathbf{v}_1 \neq \mathbf{0}) \quad \text{or} \\ &\eta = -k\xi && \text{if} \quad \mathbf{v}_2 \neq \mathbf{0} \quad \text{and} \quad \mathbf{v}_1 = k\mathbf{v}_2 \quad (k \in \mathbf{R})\ . \end{aligned}$$

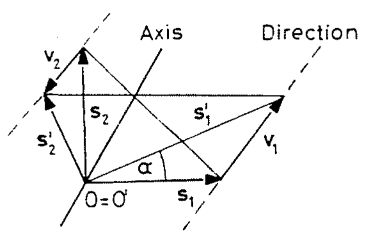

Fig. 6.

Let us give now a more detailed geometric discussion of the above classification. In each case the equations of the considered linear transformation will be given in a suitably chosen orthonormal coordinate system $\{O, \mathbf{u}_1, \mathbf{u}_2\}$.

1. We have in this class the (orthogonal) reflexions to an axis
   The equations of the transformation are

$$\begin{aligned} x' &= x, \\ y' &= cy \quad (\text{with} \quad c = \varepsilon\mu_1\mu_2 = -1), \end{aligned}$$

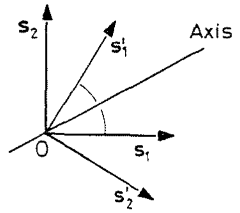

Fig. 7.

if the new base is given by

$$\begin{aligned}\mathbf{u}_1 &= \cos\frac{\alpha}{2}\mathbf{s}_1 + \sin\frac{\alpha}{2}\mathbf{s}_2\ ,\\ \mathbf{u}_2 &= -\sin\frac{\alpha}{2}\mathbf{s}_1 + \cos\frac{\alpha}{2}\mathbf{s}_2\ .\end{aligned}$$

2.1 $\cos\alpha = 1$ and $\mu_1 = 1$.
The equations of the transformation are

$$\begin{aligned}x' &= x,\\ y' &= cy \quad (\text{with } -1 < c = \varepsilon\mu_1\mu_2 < 1)\end{aligned}$$

in the base $\mathbf{u}_1 = \mathbf{s}_1$ and $\mathbf{u}_2 = \mathbf{s}_2$.

Notice that for $c = 0$ we also have a degenerated transformation (orthogonal projection onto the axis).

2.2 $\cos\alpha = 1$ and $\mu_1 \geq \mu_2 = 1$, $\varepsilon = 1$.
The equations of the transformation are

$$\begin{aligned}x' &= x,\\ y' &= cy \quad (\text{with } 1 \leq c = \varepsilon\mu_1\mu_2)\end{aligned}$$

in the base $\mathbf{u}_1 = \mathbf{s}_2$ and $\mathbf{u}_2 = -\mathbf{s}_1$.

Notice that for $c = 1$ we have here also the identity.

2.3 $\cos\alpha = -1$ and $\mu_1 > \mu_2 = 1$, $\varepsilon = -1$.
The equations of the transformation are

$$\begin{aligned}x' &= x,\\ y' &= cy \quad (\text{with } c = \varepsilon\mu_1\mu_2 < -1).\end{aligned}$$

in the base $\mathbf{u}_1 = \mathbf{s}_2$ and $\mathbf{u}_2 = -\mathbf{s}_1$.

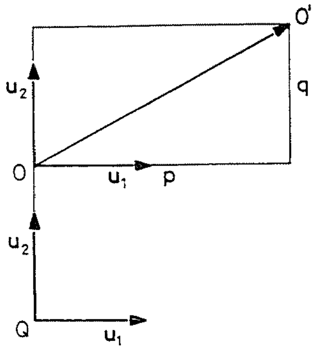

Fig. 8.

So it has been shown that any special linear transformation of classes 1. and 2. is in fact an *orthogonal axial affinity* given by the linear equations

$$x' = x,$$
$$y' = cy \quad (\text{with} \quad c \in \mathbf{R})$$

with respect to the suitably chosen orthonormal coordinate system $\{O, \mathbf{u}_1, \mathbf{u}_2\}$.

Let us consider now a special affine transformation whose linear part is an orthonormal axial affinity. Then the translation vector $\overrightarrow{OO'}$ can be written in the base $(\mathbf{u}_1, \mathbf{u}_2)$ as follows:

$$\overrightarrow{OO'} = p\mathbf{u}_1 + q\mathbf{u}_2 \; .$$

It is easy to see that the endpoint Q of the vector

$$\overrightarrow{OQ} = \frac{q}{1-c}\mathbf{u}_2 \quad (c \neq 1) \qquad\qquad (\text{see } \textit{Fig. 8})$$

should be taken for the origin of the coordinate system $\{Q, \mathbf{u}_1, \mathbf{u}_2\}$ and so the given affine transformation can easily be characterized by the following equations:

$$x' = x + p,$$
$$y' = c\,y \; ,$$

where $c = \varepsilon\mu_1\,\mu_2$ is called the ration of the affinity, and p gives the translation along the axis. (Obviously for $c = 1$ we only have a translation given by the vector $\overrightarrow{OO'}$.)

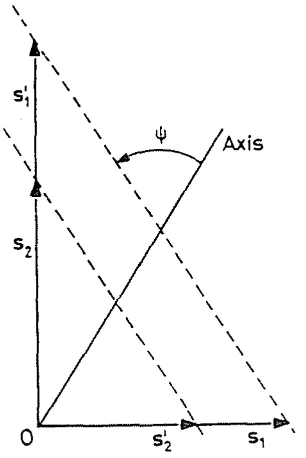

Fig. 9.

Turning back to the classification of special linear transformations we give the geometric meaning of the case 3 where $\cos\alpha = 0$ holds with $\mu_1 + \varepsilon\mu_2 > 0$. This is an oblique reflection having the axis of equation

$$\eta = \mu_1\xi = \frac{1}{\mu_2}\xi \qquad \text{if} \quad \alpha = \frac{\pi}{2},$$

or

$$\eta = -\mu_1\xi = -\frac{1}{\mu_2}\xi \qquad \text{if} \quad \alpha = -\frac{\pi}{2}$$

Posing the first vector $\mathbf{u}_1$ of the right-handed orthonormal base $(\mathbf{u}_1, \mathbf{u}_2)$ on the axis of the affinity we have the following equations (with respect to the system $\{O, \mathbf{u}_1, \mathbf{u}_2\}$):

$$x' = x + \frac{c-1}{\operatorname{tg}\psi} y,$$
$$y' = cy \quad (\text{with} \quad c = \varepsilon\mu_1\mu_2 = -1)\ ,$$

where

$$\operatorname{tg}\psi = \frac{2}{\mu_1 - \mu_2} \quad \text{if} \quad \alpha = \frac{\pi}{2},$$
$$\text{or} \quad \operatorname{tg}\psi = \frac{2}{\mu_2 - \mu_1} \quad \text{if} \quad \alpha = -\frac{\pi}{2}\ . \qquad \text{(see } \textit{Fig. 9}\text{)}$$

(ψ denotes the angle of the direction and the axis).

The description of the considered linear transformation becomes simpler if the oblique base

$$\tilde{\mathbf{u}}_1 = \mathbf{u}_1 \quad \text{and} \quad \tilde{\mathbf{u}}_2 = \cos\psi\mathbf{u}_1 + \sin\psi\mathbf{u}_2$$

is introduced. $(\psi \neq 0)$.

Then also the more general affine transformation with $\overrightarrow{OO'} = p\tilde{\mathbf{u}}_1 + q\tilde{\mathbf{u}}_2$ can be given by the following simple equations with respect to the new coordinate system $\{Q, \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2\}$

$$x' = x + p,$$
$$y' = cy \quad (\text{with} \quad c = \varepsilon\mu_1\mu_2 = -1),$$

where the origin Q is the endpoint of the vector

$$\overrightarrow{OQ} = \frac{q}{1-c}\tilde{\mathbf{u}}_2 \qquad \text{see} \quad \mathit{Fig.}\ 10$$

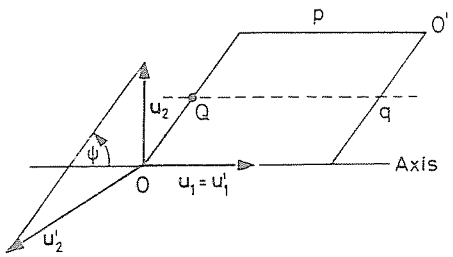

Fig. 10.

As a final step let us investigate the affine transformations whose linear part belongs to the class 4. given above.

4.0 First we give the degenerated case when

$$\varepsilon\mu_2 = 0\ , \quad \cos\alpha = \frac{1}{\mu_1} < 1\ .$$

This is evidently an oblique projection onto the axis.

If the translation vector $\overrightarrow{OO'}$ is decomposed in the oblique base

$$\tilde{\mathbf{u}}_1 = \cos\alpha\mathbf{s}_1 + \sin\alpha\mathbf{s}_2 \quad \text{and} \quad \tilde{\mathbf{u}}_2 = \mathbf{s}_2$$

$$(-\frac{\pi}{2} < \alpha < 0 \quad \text{or} \quad 0 < \alpha < \frac{\pi}{2})$$

by $\overrightarrow{OO'} = p\tilde{\mathbf{u}}_1 + q\tilde{\mathbf{u}}_2$, then in the system $\{Q, \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2\}$, where $\overrightarrow{OQ} = \frac{q}{1-c}\tilde{\mathbf{u}}_2$ we have the following simple equations of the transformation:

$$x' = x + p,$$
$$y' = cy \quad (\text{with} \quad c = \varepsilon\mu_1\mu_2 = 0).$$

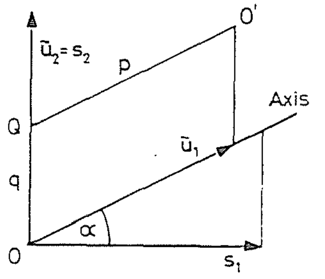

Fig. 11.

4.1 In the non-degenerated case when $\mu_1 > 1 > \mu_2 > 0$ hold we will distinguish the orientation preserving ($\varepsilon = 1$) and the orientation reversing ($\varepsilon = -1$) transformations.

In both cases simple formulas can be found for the direction of the affinity (its angle with the eigen-vector $\mathbf{s}_1$ is denoted by β) and for the axis of the affinity (its angle with the eigen-vector $\mathbf{s}_1$ is denoted by γ) (see *Fig. 6*).

For this purpose we take again the vectors

$$\mathbf{v}_1 = (\mu_1 \cos\alpha - 1)\mathbf{s}_1 + \mu_1 \sin\alpha \mathbf{s}_2 \qquad \text{and}$$
$$\mathbf{v}_2 = (-\varepsilon\mu_2 \sin\alpha)\mathbf{s}_1 + (\varepsilon\mu \cos\alpha - 1)\mathbf{s}_2 \text{ , where}$$

$$\text{where} \qquad \cos\alpha = \frac{1 + \varepsilon\mu_1\mu_2}{\mu_1 + \varepsilon\mu_2} \; .$$

Then after a short trigonometrical calculation we have

$$\mathrm{tg}^2\beta = \frac{\lambda_1(1-\lambda_2)}{\lambda_2(\lambda_1 - 1)}$$

$$\text{and} \qquad \mathrm{tg}^2\gamma = \frac{\lambda_1 - 1}{1 - \lambda_2}\,, \quad \text{where} \quad \lambda_1 = \mu_1^2 \quad \text{and} \quad \lambda_2 = \mu_2^2 \; .$$

Let us suppose now that

$$|\text{tg}\,\beta| \neq |\text{tg}\,\gamma|, \quad \text{i.e.} \quad \lambda_1\lambda_2 \neq 1\ .$$

Thus the angle $\psi = \beta - \gamma$ should not be zero here. (The case tg $\beta = -$tg γ has been considered in 3. and the case tg $\beta =$ tg γ, i.e. $\psi = 0$ will be considered in class 4.2.)

So the present class 4.1 is characterized by the conditions $\mu_1 > 1 > \mu_2 > 0$ and $\mu_1\,\mu_2 \neq 1$. Let us introduce as before the right-handed oblique base $(\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2)$ such that the unit vectors $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{u}}_2$ give the axis and the direction of the affinity, respectively. The linear transformation having the equations

$$\begin{aligned} x' &= x, \\ y' &= cy \quad (\text{with} \quad c = \varepsilon\mu_1\mu_2 \in \mathbf{R}), \end{aligned}$$

with respect to the system $\{O, \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2\}$ is called *oblique axial affinity.* Notice that our class 4.1 does not contain the cases when

$c = -1$	(oblique reflection for an axis)
$c = 0$	(oblique projection onto an axis)
$c = 1$	(identity)

(These cases have been considered in classes 3., 4.0 and 2., respectively.) If for the given special affine transformation $\mathcal{A}$ the translation vector $\overrightarrow{OO'}$ is decomposed by

$$\overrightarrow{OO'} = p\tilde{\mathbf{u}}_1 + q\tilde{\mathbf{u}}_2,$$

then in the coordinate system $\{Q, \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2\}$, with $\overrightarrow{OQ} = \dfrac{q}{1-c}\tilde{\mathbf{u}}_2$, the transformation $\mathcal{A}$ is given by the following equations:

$$\begin{aligned} x' &= x + p, \\ y' &= cy \quad (\text{with} \quad c = \varepsilon\mu_1\mu_2 \in \mathbf{R}\ , \quad c \neq 0\ , \quad |c| \neq 1) \end{aligned}$$

It should be noticed that until this point we could find for each special affine transformation $\mathcal{A}$ a suitably chosen orthonormal coordinate system $\{Q, \mathbf{u}_1, \mathbf{u}_2\}$ such that the equations of the transformation are given by

$$\begin{aligned} x' &= x + (c-1)\text{ctg}\,\psi\, y + p \quad \text{and} \\ y' &= cy\ , \end{aligned}$$

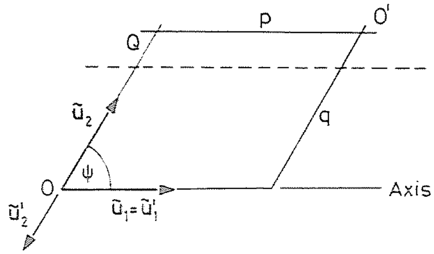

Fig. 12.

where the geometrical meaning of the 3 arbitrary real parameters c, p and ψ has been cleared in the above treatment.

4.2 The remaining class is characterized by

$$\mu_1 \mu_2 = 1 , \quad \mu_1 \neq \mu_2 \quad \text{and} \quad \varepsilon = 1 .$$

The linear part of such an affine transformation is called *elation* characterized by the fact that the axis of the affinity lies just in the direction of the affinity.

Since $\cos\alpha = \dfrac{2}{\mu_1 + \mu_2} > 0$ we have the cases

$$\sin\alpha = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} > 0 \quad \text{or} \quad \sin\alpha = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} < 0 .$$

It is easy to see that

$$\begin{aligned} \operatorname{tg}\beta &= \operatorname{tg}\gamma = \mu_1 && \text{if} \quad \sin\alpha > 0 \quad \text{and} \\ \operatorname{tg}\beta &= \operatorname{tg}\gamma = -\mu_1 && \text{if} \quad \sin\alpha < 0 . \end{aligned}$$

Then with respect to the orthonormal coordinate system $\{O, \mathbf{u}_1, \mathbf{u}_2\}$, where

$$\begin{aligned} \mathbf{u}_1 &= \cos\beta \mathbf{s}_1 + \sin\beta\, \mathbf{s}_2 , \\ \mathbf{u}_2 &= -\sin\beta \mathbf{s}_1 + \cos\beta\, \mathbf{s}_2 \quad (\beta = \gamma \quad \text{and} \quad |\beta| < \frac{\pi}{2}) \end{aligned}$$

the elation has the following equations:

$$\begin{aligned} x' &= x + (\mu_1 - \mu_2)y \quad (\text{if} \quad \sin\alpha < 0), \\ y' &= y, \\ \text{or} \quad x' &= x + (\mu_2 - \mu_1)y \quad (\text{if} \quad \sin\alpha > 0), \\ y' &= y. \end{aligned}$$

Let now

$$\overrightarrow{OO'} = p\mathbf{u}_1 + q\mathbf{u}_2$$

be the unique decomposition of the translation vector belonging to the given affine transformation. Then the origin Q of the new coordinate system should be chosen as follows:

$$\overrightarrow{OQ} = \frac{p}{\mu_1 - \mu_2}\,\mathbf{u}_2 \quad (\text{if} \quad \sin\alpha > 0)$$

$$\text{or} \quad \overrightarrow{OQ} = \frac{p}{\mu_2 - \mu_1}\,\mathbf{u}_2 \quad (\text{if} \quad \sin\alpha < 0)$$

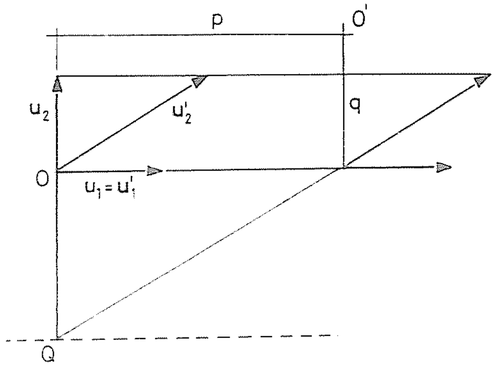

Fig. 13.

So the equations of the considered affine transformation with respect to the coordinate system $\{Q, \mathbf{u}_1, \mathbf{u}_2\}$ are given by

$$x' = x + (\mu_1 - \mu_2)\,y \text{ (if } \sin\alpha < 0),$$
$$y' = y + q,$$

or

$$x' = x + (\mu_2 - \mu_1)\,y \quad (\text{if} \quad \sin\alpha > 0),$$
$$y' = y + q.$$

This case has completed the classification of the affine transformations of the Euclidean plane.

As for the practical approach to our considerations we give the following final remark:

If the input data of any affine transformation are given by the real parameters

$$a_{11}, a_{12}, a_{21}, a_{22}, x_0, y_0$$

with respect to a standard coordinate system $\{O, \mathbf{e}_1, \mathbf{e}_2\}$, our straightforward calculations make it possible to gain the most advantageous position for a new right-handed orthonormal coordinate system. Since we need 3 output data for this purpose, in order to characterize geometrically the given affine transformation there are only at most 3 more output parameters left. It is quite clear that if there is a relation among the input data, (this is the case for example when we have to characterize a special affine transformation) then the number of independent output parameters will decrease, as well. In fact, the relation

$$\det(\mathbf{A} - \mathbf{Id}) = 0$$

implies the relation $1 + \varepsilon\mu_1\mu_2 = (\mu_1 + \varepsilon\mu_2)\cos\alpha$ among the output data μ_1, μ_2 and α, or equivalently the relation

$$\lambda_1 \cos^2\beta + \lambda_2 \sin^2\beta = \lambda_1\lambda_2$$

among the output data λ_1, λ_2 and β (see *Fig. 14*).

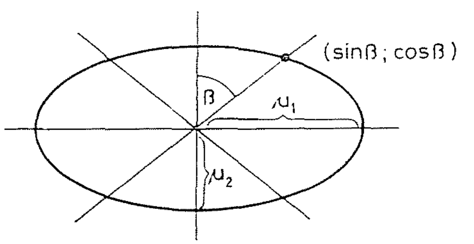

Fig. 14.

It should be evident that once the input data of an affine transformation are given a computer can always select quickly the suitable new coordinate system and the corresponding geometric parameters which appear in the reduced equations of the transformation.

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