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STABILITY OF ASYMMETRICALLY BUILT AND LOADED MULTI-LAYERED RECTANGULAR SANDWICH-TYPE PLATES (FEM SOLUTIONS)

Lajos POMÁZI and József ÚJ

Department of Applied Mechanics
Budapest University of Technology and Economics
H–1521 Budapest, Hungary
Phone: +36 1 463 2228
E-mail: pomazi@mm.bme.hu, uj@mm.bme.hu

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Abstract

The present report gives a review on the formulation of the mechanical/mathematical models of the stability of asymmetrically built and loaded rectangular multi-layered sandwich-type plates with (constructionally) orthotropic hard and transversally isotropic soft layers. The corresponding governing equations, boundary conditions and method of numerical (FEM) solution are given. The report shows the stability and post-buckling behaviour of asymmetrically compressed five-layered rectangular plate with Navier-type boundary conditions.

Keywords: stability, post-buckling, asymmetric sandwich plate, finite element method.

1. Introduction, Formulation of the Task

In the theory of structures with combined materials the sandwich-type ones play special role because of their good and predictable physical characteristics. Theories of classical three and regularly multi-layered sandwich plates and shells are done well in the literature. From the point of view of practical applications irregularly built multi-layered plate and shell structures are also very important. Our investigations are dealing with such structures, closely the aim of them are the comparative study of the analytical, experimental and numerical solution of the stability for asymmetrically built and loaded multi-sandwich rectangular plates with (constructionally) orthotropic hard and transversally isotropic soft layers, using the common suppositions in the theory of sandwich and sandwich-type structures (HOFF (1950), BOLOTIN (1965), SUN et al. (1968)).

According to these suppositions for the transversally isotropic soft layers not only the transverse shear strains and stresses but the normal ones are also constant across these layers and taken into account. Following the formalism of BOLOTIN (1965) and neglecting the second order terms the strain *energy densities* of hard and transversally soft layers are respectively:

$$\overline{u} = \frac{1}{2}\left(\overline{\sigma}_x\overline{\varepsilon}_x + \overline{\sigma}_y\overline{\varepsilon}_y + \overline{\tau}_{xy}\overline{\gamma}_{xy}\right),$$
$$\tilde{u} = \frac{1}{2}\left(\tilde{\tau}_{xz}\tilde{\gamma}_{xz} + \tilde{\tau}_{yz}\tilde{\gamma}_{yz} + \tilde{\sigma}_z\tilde{\varepsilon}_z\right).$$

Here the term $\tilde{\sigma}_z\tilde{\varepsilon}_z$ corresponds to the effect of anti-plane tension /compression of the transversally soft layer. If this term is negligibly small compared with others, then the layer is as usual *soft*. More sophisticated analysis of the characters of the layers is given by BOLOTIN (1965), POMÁZI and MOSKALENKO (1967).

Our former investigations for the stability of regularly multi-layered (POMÁZI (1974 and 1992)) and asymmetrically built and loaded three-layered plates with constructionally orthotropic hard layers (POMÁZI (1980, 1985 and 1990)) give the basic methods, formulas and governing equations of the problem. The mechanical-mathematical model of a multi-layered plate with n hard layers is based on the displacement field of the plate, which could be described by $3n$ displacement functions (as Lagrangean coordinates) in the coordinate directions $u_\alpha(x, y)$, $v_\alpha(x, y)$, $w_\alpha(x, y)$ of the points belonging to the basic surfaces of the hard layers. The displacements in the hard layers are determined by the Kirchhoff–Love law; in the soft layers – based on the suppositions – they are linear functions of local normal coordinates and can be expressed by the displacements of adjacent hard layers. So, using this linearity of displacement functions and the corresponding constitutive equations the strain and stress fields of the hard and soft layers can be obtained (POMÁZI (1980 and 1992)). With the integration of the deformation energy densities (1) for unit surface and using the generalized constitutive equations the *total potential energy of the system for the state of small disturbances* – $I\langle u_\alpha, v_\alpha, w_\alpha\rangle$ – can be written. For the mathematical formulation of the stability problem the Trefftz variational principle $\delta(\delta_*^2 U_0) = 0$ can be used, where in this case $\delta_*^2 \cdot U_0 = I$, i.e. the second special variation of the total potential energy of the system for the disturbance-less state is equal to the total potential energy of the system for the state of small disturbances. So, the governing equations and natural boundary conditions have been derived as Euler–Lagrange equations and boundary conditions to the $\delta I\langle U_\alpha, v_\alpha, w_\alpha\rangle = 0$ variational principle.

For the stability of multi-layered plate with n hard layers system of governing equations consists of $3n$ difference-differential equations, expressing that the mechanical-mathematical model of the problem is continuous by the in-plane coordinates x, y and discrete by the coordinate z, perpendicular to the plate, expressed by label: α. (The difference $\Delta\alpha$ depends on the place of the $[x, y]$ plane of coordinates and usually $\Delta\alpha = 1$, i.e. the label of the layers begins from the first one and $1 \leq \alpha \leq n$, but if the plate is symmetric to the middle hard layer, then $\Delta\alpha = 2$ and $-m \leq \alpha \leq m$ where $m = n - 1$ (POMÁZI (1974)). These equations and boundary conditions with the solution of them for the regularly multi-layered rectangular plate are given by POMÁZI (1992).

The natural boundary conditions at $x =$ const. are:

1. $[C_{11}\varepsilon_x + C_{12}\varepsilon_y + K_{11}\kappa_x + K_{12}\kappa_y]_\alpha = N_x^\alpha = 0,$
2. $C_{66}^\alpha \gamma_\alpha + 2K_{66}^\alpha \chi_\alpha = N_{xy}^\alpha = 0,$
3. $-r'_\alpha \tilde{\tau}_{xz}^\alpha - N_x^\alpha w_{\alpha,x} - N_{xy}^\alpha w_{\alpha,y} + M_{x,x}^\alpha + M_{xy,y}^\alpha = 0,$
4. $-[K_{11}\varepsilon_x + K_{12}\varepsilon_y + D_{11}\kappa_x + D_{12}\kappa_y]_\alpha = M_x^\alpha = 0,$
5. $-[K_{66}\gamma + 2D_{66}\chi]_\alpha = M_{xy}^\alpha = 0.$

The last condition expresses vanishing of the twist moment on the boundary, however, according to Kirchhoff this does not mean a new condition, but the derivative of it gives an additional shearing force in the 3rd condition. So, in spite of the fact, that in case of $K_{ik} \neq 0$ order of the governing equations by the in-plane coordinates increases to $3 + 3 + 4 = 10$, the number of condition in a certain point of the boundary remains 4 and not 5 as it could be supposed. Proving of this in detail is given by POMÁZI (1996).

2. About the Analytical Solution of the Governing Equations at Navier-Type Boundary Conditions

The Navier-type boundary conditions mean that the normal displacements and bending moments are equal to zero at the boundary of each hard layer (they are quasi simple supported on the edges). This condition can be physically modelled by fixing the hard layer edges to a membrane in the plane of the boundary, absolutely rigid at in-plane, but absolutely soft by out of plane deformation. According to the loading of the plate it is supposed, that the hard layers could be loaded with constant normal to the boundary membrane forces N_x^α, N_y^α and at least one of the hard layers is loaded, but the shearing membrane forces – N_{xy}^α – are equal to zero. The sum of these normal forces gives the total loading of the plate: $N_x = \sum_{(\alpha)} N_x^\alpha$, $N_y = \sum_{(\alpha)} N_y^\alpha$ and their values are proportional to the stiffness of the loaded hard layers (the total side-loads are distributed to the loaded layers according to the stiffness of each). In this case these boundary conditions will be satisfied if – using the Fourier method – the unknown displacement functions are taken in double Fourier series:

$$u_\alpha = \sum_{p_1} \sum_{p_2} U_\alpha \cos k_1 x \sin k_2 y,$$
$$v_\alpha = \sum_{p_1} \sum_{p_2} V_\alpha \sin k_1 x \cos k_2 y,$$
$$w_\alpha = \sum_{p_1} \sum_{p_2} W_\alpha \sin k_1 x \sin k_2 y,$$

where U_α, V_α, W_α are amplitude functions, $k_1 = p_1\pi/a = \pi\lambda_x$, $k_2 = p_2\pi/b = \pi/\lambda_y$ are the wave numbers, p_1, p_2 are whole numbers, a, b are the side lengths of the plate, λ_x, λ_y are the half wave lengths at bending of the plate. As usual, substituting the trial solution functions into the governing equations, from the conditions of existence for a non-trivial solution to the amplitude functions we get the characteristic equation of the problem in the form:

$$\det(\underline{\mathbf{A}} - N_\alpha(\underline{\mathbf{I}}_3 - \underline{\mathbf{I}}_2)) = 0, \tag{1}$$

where $\underline{\mathbf{A}}$ is the coefficient matrix of the eigenvalue problem, $\underline{\mathbf{I}}_3$, $\underline{\mathbf{I}}_2$, are the 3×3 and 2×2 idem matrices and $N_\alpha = k_1^2 N_x^\alpha + k_2^2 N_y^\alpha$ is the '*loading parameter*'. The matrix $\underline{\mathbf{A}}$ is diagonal strip matrix consisting of 3×3 submatrices. The minimal eigenvalue of the characteristic equation gives the critical value of the loading parameter: $\min N_\alpha \stackrel{*}{=} N_{\text{crit}}^\alpha$. Before the solution to the governing equation the asymmetry of structure and loading of the plate have to be examined, to determine the conditions of the correct formulation of the stability problem.

In the case of constructionally anisotropic asymmetric hard layers ($K_{ik} \neq 0$) at an optional chosen basic surface there is a coupling between the stretching and bending, so the in-plane deformation of the plate before buckling can be achieved with assuring some special loading circumstances only. Such '*bendingless*' state of the plate could be achieved by choosing new basic surfaces for the hard layers from the condition that the curvatures of these new surfaces should be equal to zero: $\kappa_{ik} = 0$. The corresponding formulas are given by POMÁZI (1990). In the case of very different anisotropy in x, y directions, the $\kappa_{ik} = 0$ condition cannot be satisfied by a common new basic surface in both directions, but in this case it could be taken by the ratios of the stiffness characteristics at unidirectional tension in the x and y directions: $A_{11}^\alpha / A_{22}^\alpha$.

The detailed analytical solution of the governing equations with Navier-type boundary conditions for case of asymmetrical building and loading condition is given by POMÁZI (1996).

3. Formulation of the FEM Model

The mechanical and mathematical model of the stability investigations in the previous section is based on the shell/plate theory: hard layers are regarded as thin Kirchhoff–Love plates. The mechanical state in the soft layers is supposed to be determined by the deformation of the adjacent hard layers. As a special consideration, the out off plane normal stress and strain are taken into account in the soft layers. This way the governing equations and the boundary conditions are based on the shell theory.

The finite element method gives excellent opportunities to overtake some simplifying considerations of the analytical model. First of all, the shell/plate theory is not considered, that is in the FEM model the soft and hard layers are three dimensional objects meshed by the help of solid elements. Additionally, the FEM model does not apply more coupling effect between the hard and soft layers than only one: the perfect adhesion of contacted soft and hard surfaces is supposed.

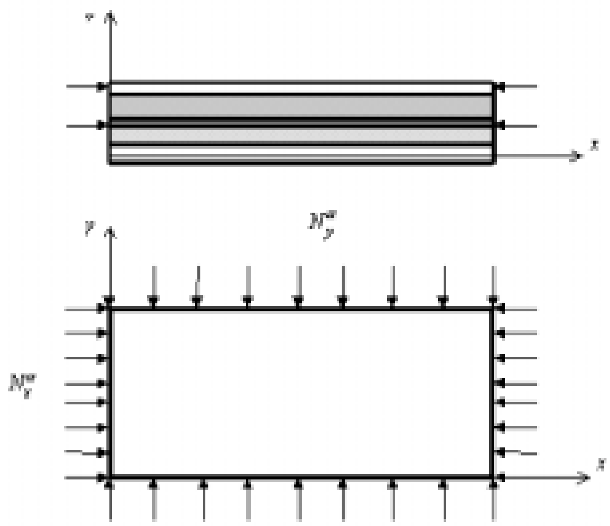

Fig. 1. Asymmetrically built and loaded multi-sandwich plate

According to these considerations, the FEM model of the 5-layer sandwich plate consists of 5 volume domains (each for one layer), that are meshed by 8-node solid elements. COSMOS/M 1.75 Explorer software was applied for the buckling analysis. The kinematic boundary conditions represent pin-type constraints along the long faces of the hard layers, that is the eligible displacement coordinates are supported to be zero along those edges. The axial load was given as surface pressure acting on the faces $x = a$ of the hard layers (*Fig. 2*). A buckling analysis using Lanczos algorithm was applied to determine the first eigenvalue and eigenmode. The number of elements was 21 along x and y directions and transversally 1 element was applied for each layer. In the Explorer version the number of elements is limited to 3000.

The finite element method also gives opportunities to study the post-buckling behavior of the 5-layer sandwich plate. The post-buckling FEM model was the same as for buckling analysis, but the number of elements was reduced to spare the analysis time. In post-buckling analysis the axial compressive load was controlled by a load-time curve (*Fig. 3*). The compressive load was chosen to exceed the first buckling load by 20%. At the same time, two transversal point loads with separate load-time curve were also applied to ensure the disturbed deformation of the sandwich plate. These two transversal point loads generate a deflected shape close to the first buckling mode shape. Non-linear analysis was applied for post-buckling analysis, force control technique and Total Lagrange formulation was chosen. 50 time steps were applied for the total load process.

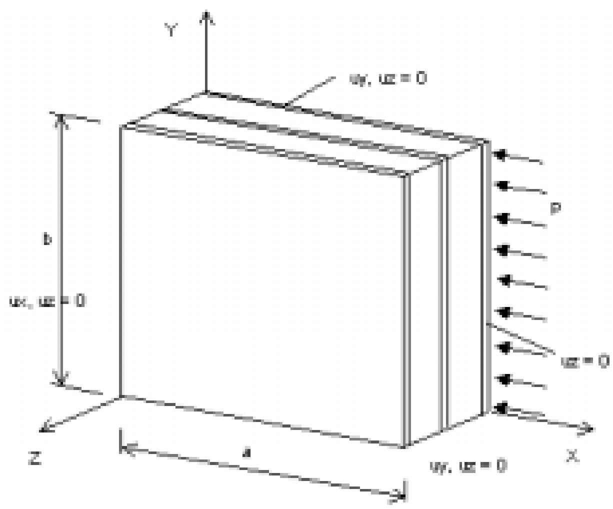

Fig. 2. Volume domains and boundary conditions for the FEM analysis of buckling

4. Numerical Investigations

Using the elaborated FEM model the stability of asymmetrically built and loaded rectangular sandwich plate with 5 hard layers was investigated. The calculation was executed by using the COSMOS'M FEM program. The main aim of the investigations was to give basic data to the comparative study of analytical, experimental and FEM-based investigations, as it was mentioned. Therefore, the calculations basically were made for the structure and measurements data of the experiments specimens (*Table 1*). Material characteristics of these specimens are for the hard layers: Fe (Steel): $E = 2.0\ 10^4$ kN/cm^2, $\nu = 0.3$; Al (Aluminum): $E = 0.68\ 10^4$ kN/cm^2, $\nu = 0.3$; and for the transversally soft layers (matrix): PUR foam: $E_z = E_z = 0.18$ kN/cm^2, $\nu = 0.125$.

Table 1. Structure and measurement data (in mm) of the specimens

	No. 1.		No. 2.		No. 3.		No. 4C.		No. 5.	
s1	10		10		10		10		10	
s2	15		15		25		25		25	
h1	1.2	Al	1.0	Al	1.0	Al	0.8	Fe	1.5	Al
h2	1.0	Fe	1.0	Fe	0.8	Fe	1.5	Al	1.0	Al
h3	1.2	Al	1.2	Al	1.0	Al	1.0	Al	1.0	Al

In *Table 2* the results for the specimen 4C, based on the analytical model are given for the case of unidirectional in-plane loading of the hard layers, i.e. when

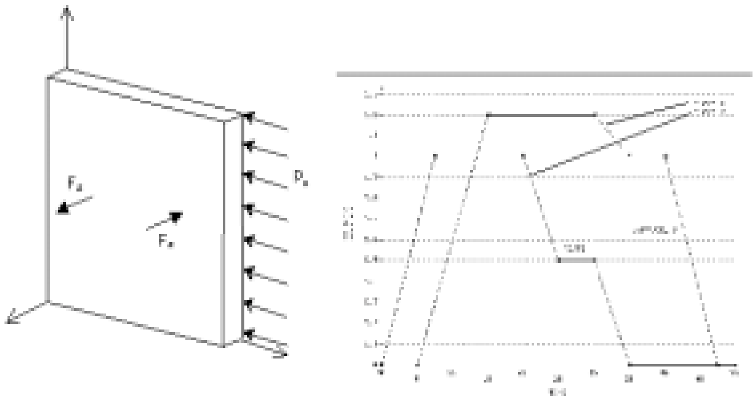

Fig. 3. Axial and transversal loads in the post-buckling analysis with the load-time curves

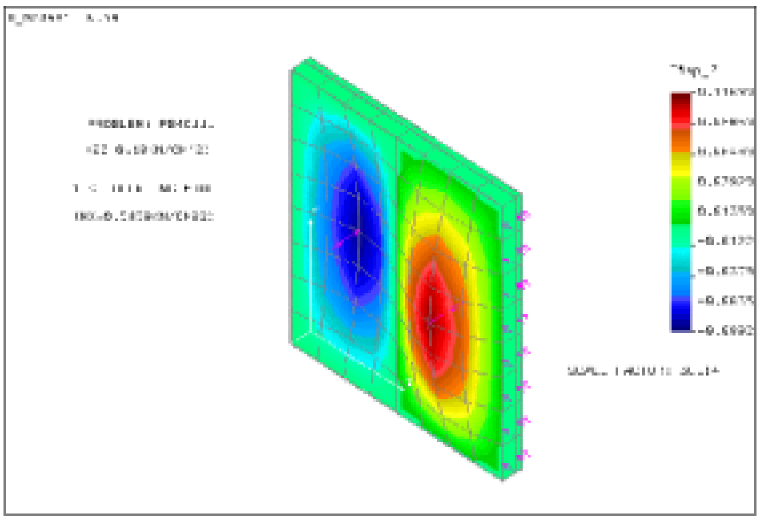

Fig. 4. 1-st buckling mode of the plate

$N_y/N_x = 0$. In this Table the number 4C is related to the specimen investigated in detail and the row-vectors (as 'code') in column 2 give the *mode of loading* i.e. in order 1,2,3 of hard layer show if the layer is loaded (1) or not (0). It was a little-bit surprising that for the different mode of loading – at the given material data – the character and critical values of the first buckling mode were very similar in spite of the strong asymmetry of the structure and loading of the plate. Going out of these circumstances in the following our calculations will be shown for the case of loading (1,1,1), when all the three hard layers are loaded.

The linear stability investigations for the 1-st buckling mode gave that the

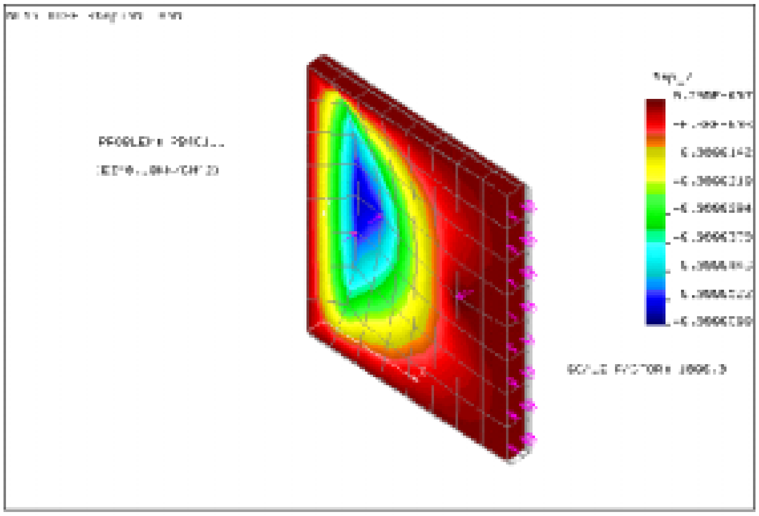

Fig. 5. Normal to the plate U_z displacement field in the post-buckling state

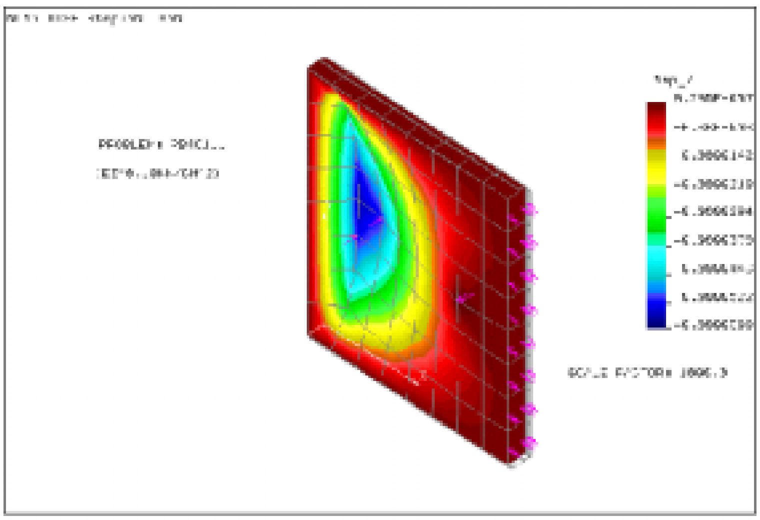

Fig. 6. The U_z displacements of the normal to the y-axis middle cross section

half-wave lengths of the instability correspond to $p_1 = 2$, $p_2 = 1$ wave numbers, i.e. in the direction of loading two half waves occur (*Fig. 4*). This means that the instability of the plate has 'local' character. It was found that this character is valid for all the load 'codes'. The reason of this is that the normal-to-the-plate stiffness of the soft layers is very small, but big enough to keep together the hard layers for the similar deformation.

Table 2. Critical loads (kN/cm) of five-layered sandwich-type plate ($E_z = 0.18$ kN/cm^2)

Code of specimen	Code of loading	$\sum N_x$	N_x^1	N_x^2	N_x^3	Method of solution
4C	1 1 1	0.5233	0.2537	0.1617	0.1078	Analytical
	1 1 0	0.5136	0.3136	0.2000	0	
	1 0 1	0.5179	0.3634	0	0.1545	
	0 1 1	0.5254	0	0.3152	0.2102	
	1 0 0	0.4982	0.4982	0	0	
	0 1 0	0.5187	0	0.5187	0	
	0 0 1	0.4877	0	0	0.4877	
	1 1 1	0.544	0.264	0.168	0.112	FEM COSMOS'M
	1 1 0	0.533	0.326	0.207	0	
	1 0 1	0.537	0.377	0	0.160	
	0 1 1	0.550	0	0.330	0.220	
	1 0 0	0.514	0.514	0	0	
	0 1 0	0.543	0	0.543	0	
	0 0 1	0.526	0	0	0.526	

The initial post-buckling state of the plates was investigated in 'nonlinear' sense up to the value of overloading parameter $N_x/N_x(\text{crit}) = 1.2$. The characteristic picture of the normal displacement field and the deformed shape of the normal to the y-axis middle cross-section are shown on the *Fig. 5* and *Fig. 6*. For all the load 'codes' the intensive normal deformation (like local 'bending') of the bottom range was characteristic, but the shape of other parts of the plate was different according to the modes of loading. During the nonlinear task the normal displacements of the nodes at the place of disturbing F_z forces (*Fig. 3*) and the nodes at the middle points of loaded edges were monitored on the bottom (nodes 363, 367, 369) and on the top (nodes 444, 448, 450) of the plate. These displacements are shown in the *Fig. 7*, where positive values belong to the U_z normal displacements of nodes 367, 448; negative values to same displacements of the nodes 363, 444, and the 2 middle curves – in enlarged form in the *Fig. 8* – show the U_x axial displacements of nodes 369, 450 of the loaded surface. These last ones are the shortening of the 1-st and 5-th hard layers. From these figures one can see that in spite of the very soft layers the normal deformation (changing of the thickness) of the plate remains small.

Fig. 9 shows the U_z normal displacement of node 363 and the axial one of node 369 both belonging to the 1-st hard layer on the bottom surface of the plate in the beginning range of loading (up to the half of the critical load). Comparing these curves with the load – time curves in the *Fig. 3*, it can be seen that up to the time $T = 5$ the F_z disturbing forces cause small normal displacements, which remain constant up to the time $T = 8$, when this displacement begins to grow due to the growing of N_x axial load. In the range of loading between the two time values mentioned above the plate is (more or less) in 'bendingless' state and the 'instability' at the time $T = 8$ has begun when the load's value is only about 60% of the critical one. This is one of the consequences of the asymmetric building and loading of the plate. This can also be seen in the *Fig. 10* , which shows the nonlinear function of the overloading parameter on the plate's shortening.

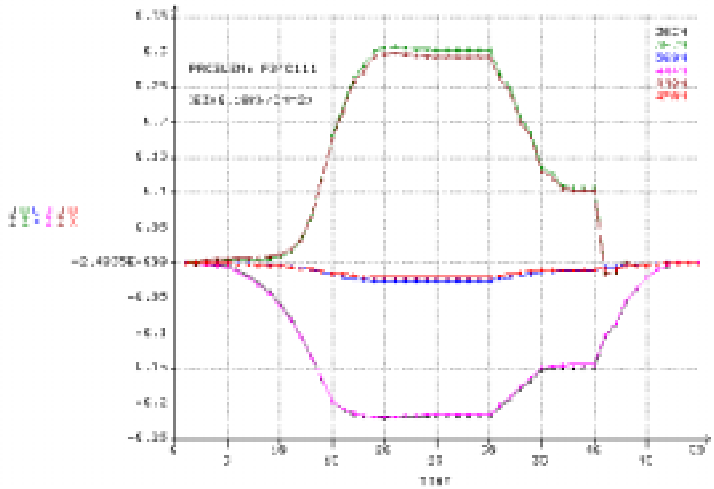

Fig. 7. Displacements – time curves of the given nodes

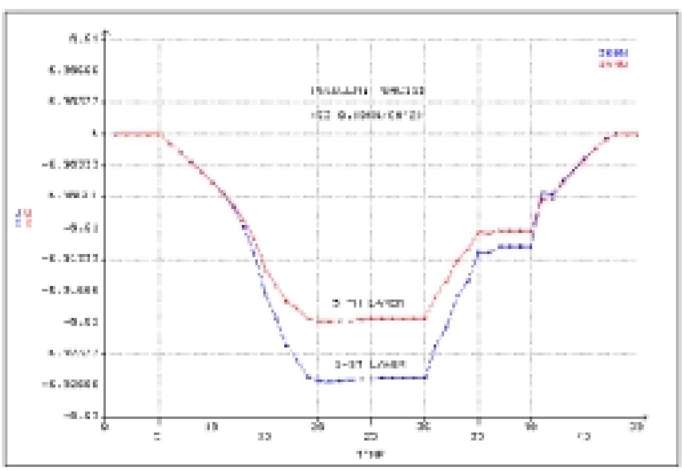

Fig. 8. The shortening (U_x) – time curves of the 1-st and 5-th loaded hard layers

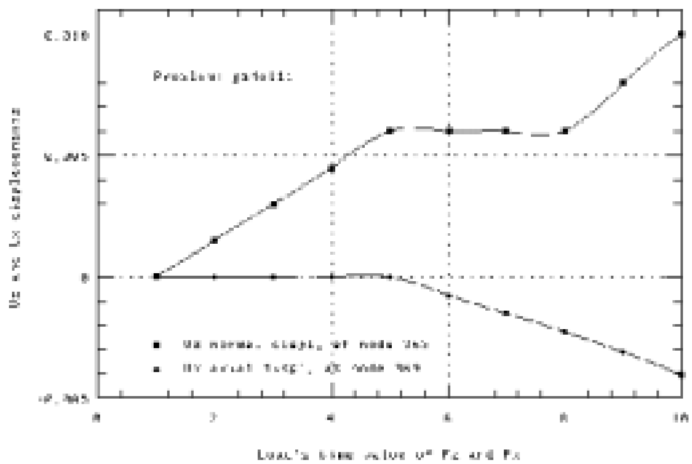

Fig. 9. U_z and U_x displacements at the beginning range of loading

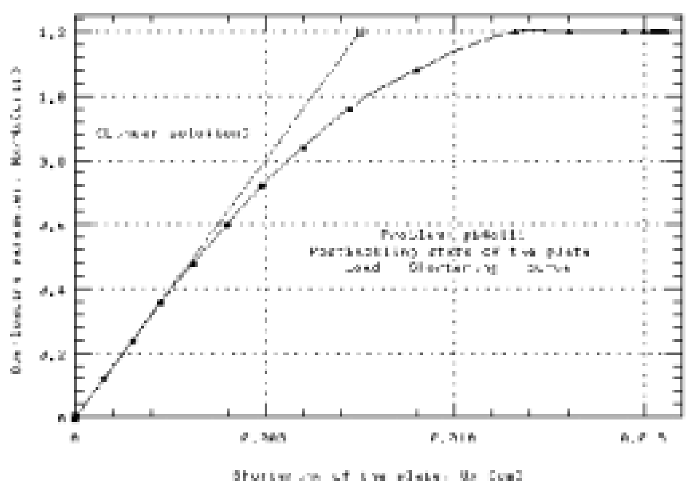

Fig. 10. The load – shortening curve of the plate

5. Conclusions

Numerical results show good agreement with those got on the base of an analytical model, but also show that because of strong asymmetry, these results are close to each other in spite of the different structure of the plate. It is also shown that changing of the thickness (normal deformations) is small contrary to all expectations. The form of instability has 'local' character, i.e. the critical half-wave length is less than the side length of the plate ($p_2 > 1$). In the frame of our research we found that by increasing of the soft layer's Young modulus the critical half-wave length is decreasing. For example, if in the case shown above the Young modulus increases 10 times to the value 1.8 kN/cm^2, then $p_1 = 1$, $p_2 = 4$, i.e. the critical half-wave length = $L_x/4 = 10.5$ cm. All these circumstances need more investigations and the results will be published in the near future.

Acknowledgement

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