PENDULUM WITH HARMONIC VARIATION OF THE SUSPENSION POINT

Tamás INSPERGER and Roland HORVÁTH

Technical University of Budapest
Department of Applied Mechanics
H–1521 Budapest, Hungary

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Abstract

A conventional pendulum has two equilibria, the lower one, and the upper one. This paper presents the stability problems of the upper equilibrium state in case of parametric excitation. We will show that the upper equilibrium can be stable due to the harmonic variation of the pendulum suspension point. By manufacturing the pendulum and the oscillator, we proved the theoretical results in practice.

Keywords: parametric excitation, Floquet theory.

1. Introduction

An oscillatory system with parametric excitation means that some of the parameters of the system change as a periodic function of time. This phenomenon can be used for stabilizing basically unstable processes. For example unstable cutting can be stabilized by periodic variation of the cutting speed, or turbulent flows can become laminar by periodic spurt of some fluid into the flow.

In this paper, we will examine how the upper equilibrium point of the pendulum can be stabilized with parametric excitation. In this case, the parametric excitation means the harmonic variation of the point of suspension. The main idea is to show how we can eliminate an oscillation with the help of another oscillation.

2. Mechanical Model of the Pendulum

Nomenclature:

- m: mass of the rod
- S: the center of gravity of the rod
- l_S: distance between point of suspension and center of gravity S
- φ: angular displacement of the rod
- Θ_S: mass moment of inertia of the rod
- r: amplitude of the oscillation
- ω: angular frequency of oscillation

This model disregards the damping effect produced by friction and air resistance.

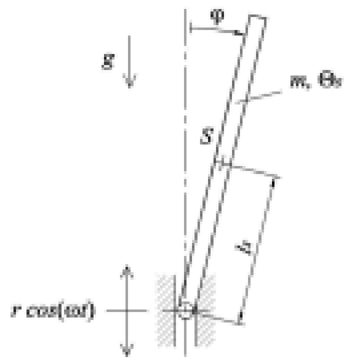

Fig. 1. Mechanical model of the pendulum

First, we consider a pendulum without parametric excitation, that is $r = 0$. In this case the equation of motion assumes the form:

$$\left(ml_S^2 + \Theta_S\right)\ddot{\varphi} - mgl_S \sin\varphi = 0. \tag{1}$$

This is a nonlinear ordinary differential equation. Substituting $\varphi \equiv \varphi_0$, $\dot{\varphi} \equiv 0$ and $\ddot{\varphi} \equiv 0$ we get the equilibrium points of the system. In practice there are two equilibrium points, the upper one ($\varphi_0 \equiv 0$), and the lower one ($\varphi_0 \equiv \pi$). We are interested in the upper equilibrium. Linearizing the *Eq.* (1) at the position $\varphi_0 \equiv 0$, we get

$$\ddot{\varphi} - \frac{mgl_S}{\left(ml_S^2 + \Theta_S\right)}\varphi = 0.$$

The necessary and sufficient condition of stability for these kinds of equations is that the coefficient of φ is nonnegative. In this case, this condition is not satisfied, that is the upper equilibrium is unstable.

Now, take also the parametric excitation into account. The equation of motion has the form:

$$\left(ml_S^2 + \Theta_S\right)\ddot{\varphi} + \left(-mgl_S + mr\omega^2 l_S \cos(\omega t)\right)\sin\varphi = 0 \tag{2}$$

This system has the same equilibrium points, the upper one ($\varphi_0 \equiv 0$), and the lower one ($\varphi_0 \equiv \pi$). Linearization of *Eq.* (2) at position $\varphi_0 \equiv 0$ results:

$$\frac{1}{\omega^2}\ddot{\varphi} + \left(-\frac{mgl_S}{\left(ml_S^2 + \Theta_S\right)\omega^2} + \frac{mrl_S}{ml_S^2 + \Theta_S}\cos(\omega t)\right) = 0. \tag{3}$$

This is a linear ordinary differential equation with a periodically time dependent coefficient. If there is no parametric excitation, that is $r = 0$, then this equilibrium point is unstable, as we know from practice and as shown above. The question is whether there are such r, ω amplitude-frequency pairs, that the equilibrium becomes stable.

3. Stability of Homogeneous Linear Systems with Periodic Coefficients

In this section we consider the general linear system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{4}$$

where the coefficient matrix $\mathbf{A}(t)$ is periodic with period $T > 0$, that is $\mathbf{A}(t+T) = \mathbf{A}(t)$. We are interested in the stability of the equilibrium point $\mathbf{x} \equiv 0$. To solve the problem, we apply the Floquet theorem (FARKAS, 1994 and HIRSCH – SMALE, 1974).

The fundamental matrix of (4) is $\Phi(t)$, if the

$$\frac{d\Phi(t)}{dt} = \mathbf{A}(t)\Phi(t)$$

matrix differential equation is satisfied. The following statements can be proved:

- All solutions of (4) can be written in the form $\Phi(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector.
- There exists a fundamental matrix $\Phi_0(t)$, that all solutions of (4) come up in the form $\Phi_0(t)\mathbf{x}_0$, where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial condition, that is $\Phi_0(0) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.
- All fundamental matrices can be written in the form $\Phi(t)\tilde{\mathbf{C}}$, where $\tilde{\mathbf{C}}$ is a constant matrix.
- For any fundamental matrix $\Phi(t)$, $\Phi(t+T)$ is also a fundamental matrix.
- There exists constant matrix $\tilde{\mathbf{C}}$ for which $\Phi(t+T) = \Phi(t)\tilde{\mathbf{C}}$, where $\tilde{\mathbf{C}}$ is called the principal matrix of (4), $\tilde{\mathbf{C}} = \Phi^{-1}(t)\Phi(t+T)$.
- The principal matrix belonging to the fundamental matrix $\Phi_0(t)$ assumes the form $\mathbf{C} = \Phi_0^{-1}(t)\Phi_0(t+T) = \Phi_0^{-1}(0)\Phi_0(0+T) = \Phi_0(0)$.
- All principal matrices are similar to each other, consequently the eigenvalues of the principal matrix – called the characteristic multipliers (notation: λ_1, $\lambda_2, \ldots\lambda_n$) – are invariant, and determined by the system.
- System (4) is asymptotically stable if and only if $|\lambda_i| < 1$, $i = 1, 2, \ldots n$.
- System (4) is stable in the Liapunov sense if and only if $|\lambda_i| \leq 1$, $i = 1, 2, \ldots n$, and if $|\lambda_i| = 1$, then λ_i is simple in the minimal polynomial of the system.

In general the principal matrix cannot be determined in an analytic way, but there are several methods to approximate it (SINHA – WU, 1991). If the coefficient matrix $\mathbf{A}(t)$ is piecewise constant, then – by the coupling of solutions – the correct solution at time $t = T$ is obtained in the form:

$$\mathbf{x}(T) = \exp\left(t_n\mathbf{A}_n\right)\exp\left(t_{n-1}\mathbf{A}_{n-1}\right)\ldots\exp\left(t_1\mathbf{A}_1\right)\mathbf{x}_0,$$

$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1 & \text{if} \quad 0 \leq t \leq t_1 \\ \mathbf{A}_2 & \text{if} \quad t_1 < t \leq t_1 + t_2 \\ \vdots & \quad\ \vdots \\ \mathbf{A}_n & \text{if} \quad t_1 + t_2 + \cdots + t_{n-1} < t \leq t_1 + t_2 + \cdots + t_n = T \end{cases}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, …, $\mathbf{A}_n$ are constant matrices forming the piecewise constant $\mathbf{A}(t)$. So the principal matrix takes the form:

$$\mathbf{C} = \exp\left(t_n \mathbf{A}_n\right) \exp\left(t_{n-1} \mathbf{A}_{n-1}\right) \dots \exp\left(t_1 \mathbf{A}_1\right). \tag{5}$$

If the coefficient matrix $\mathbf{A}(t)$ is not piecewise constant, then we can replace its elements by piecewise constant values in the following way:

$$\tilde{\mathbf{A}}(t) = \begin{cases} \mathbf{A}_1 = \mathbf{A}\left(\frac{T}{2n}\right) & \text{if} \quad 0 \le t \le \frac{T}{n} \\ \vdots & \vdots \quad \vdots \\ \mathbf{A}_k = \mathbf{A}\left(\frac{T}{2n}(2k-1)\right) & \text{if} \quad \frac{T}{n}(k-1) \le t \le \frac{T}{n}k \\ \vdots & \quad \vdots \\ \mathbf{A}_n = \mathbf{A}\left(\frac{T}{2n}(2n-1)\right) & \text{if} \quad \frac{T}{n}(n-1) \le t \le T \end{cases}$$

Matrix $\tilde{\mathbf{A}}(t)$ is also periodic with period T. Applying (5) to $\mathbf{A}_1$, …, $\mathbf{A}_k$, …, $\mathbf{A}_n$ matrices and $t_1 = t_2 = \dots t_n = T/n$ time intervals, we obtain an approximation of the principal matrix. By examining the eigenvalues, we can approximately determine the stability of system (4). The bigger n is, the more correct the approximation is.

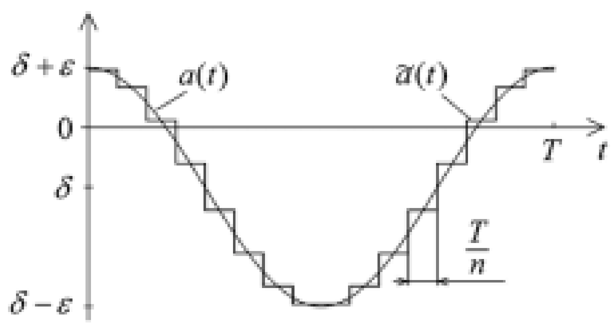

Fig. 2. Piecewise approximation

4. Stability of the Upper Equilibrium of the Pendulum

Now, examine original pendulum problem with *Eq.* (3). Introducing the new variables (by abuse of notation with respect to t):

$$\delta = -\frac{mgl_S}{\left(ml_S^2 + \Theta_S\right)\omega^2}, \qquad \varepsilon = \frac{mrl_S}{ml_S^2 + \Theta_S}, \qquad t = \omega t, \tag{6}$$

we get a simple form of differential equation, called Mathieu's equation:

$$\ddot{x} + (\delta + \varepsilon \cos t)x = 0.$$

We search for a stability map, which shows the stability in the plane of the two parameters, δ and ε of Mathieu's equation. By Cauchy transformation, we get the following system:

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y},$$

$$\mathbf{y} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \qquad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix}, \qquad a(t) = \delta + \varepsilon \cos t.$$

Matrix $\mathbf{A}(t)$ can be approximated in the way shown before:

$$\tilde{\mathbf{A}}(t) = \begin{bmatrix} 0 & 1 \\ -\tilde{a}(t) & 0 \end{bmatrix},$$

where $\tilde{a}(t)$ is a piecewise approximation of $a(t) = \delta + \varepsilon \cos t$ (see *Fig. 2*):

$$\tilde{a}(t) = \begin{cases} \delta + \varepsilon \cos\left(\frac{2\pi}{2n}\right) & \text{if} \quad 0 \le t < \frac{2\pi}{n} \\ \delta + \varepsilon \cos\left(\frac{2\pi}{2n}3\right) & \text{if} \quad \frac{2\pi}{n} \le t < \frac{2\pi}{n}2 \\ \vdots & \vdots \quad \vdots \\ \delta + \varepsilon \cos\left(\frac{2\pi}{2n}(2k+1)\right) & \text{if} \quad \frac{2\pi}{n}(k-1) \le t < \frac{2\pi}{n}k \\ \vdots & \vdots \quad \vdots \\ \delta + \varepsilon \cos\left(\frac{2\pi}{2n}(2n+1)\right) & \text{if} \quad \frac{2\pi}{n}(n-1) \le t < 2\pi \end{cases}$$

Applying (5), we get the principal matrix. By the examination of the characteristic multipliers, the stability map, the so-called Incze–Strutt diagram comes up (*Fig. 3*).

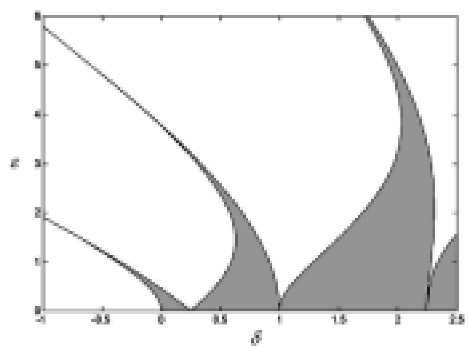

Fig. 3. Incze–Strutt diagram

To transform this map into the $r - f$ plane – where $f = \omega/2\pi$ is the frequency of the oscillation – we need the technological parameters of the pendulum. The natural angular frequency in the neighborhood of the lower equilibrium of the pendulum without parametric excitation has the form:

$$\alpha^2 = \frac{mgl_S}{ml_S^2 + \Theta_S} = \left(\frac{2\pi}{T}\right)^2,$$

where T is the period of oscillation. Substituting this into (6), we obtain

$$\delta = -\frac{mgl_S}{\left(ml_S^2 + \Theta_S\right)\omega^2} = -\left(\frac{\alpha}{\omega}\right)^2 = -\left(\frac{2\pi}{T\omega}\right)^2 = -\left(\frac{1}{Tf}\right)^2,$$

$$\varepsilon = \frac{mrl_S}{ml_S^2 + \Theta_S} = \alpha^2 \frac{r}{g} = \left(\frac{2\pi}{T}\right)^2 \frac{r}{gf},$$

that is, the parameters δ, ε can be expressed as a function of the time period T.

In practice we manufactured a pendulum with $T = 0.366$ [s]. The stability map of this pendulum in the $r - f$ plane can been seen in *Fig. 4.* The oscillator worked with the following parameters:

– frequency: $f = 36.75$ [Hz],

– amplitude: $r = 4.085$ [mm].

This point is marked with a cross on the stability map.

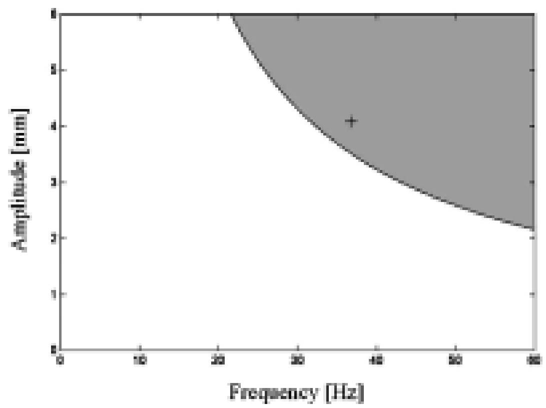

Fig. 4. Stability map

We took some photos about the pendulum oscillating with the $r - f$ amplitude-frequency pairs given above (see *Figs 5* and *6*). We can see that the rod of the pendulum is standing vertically, that is the upper equilibrium state is stable. The pendulum lighted with stroboscope can be seen in *Fig. 6.*

5. Conclusions

The stability theorem of homogeneous linear systems with periodic coefficients is well known in mathematics, but due to its complexity, it is rarely applied in industrial problems, in practical life. In this paper we described the main points of the Floquet theorem, and showed in practice how it works. The pendulum is just one – but very spectacular – possibility of applying the theorem. There are several other fields

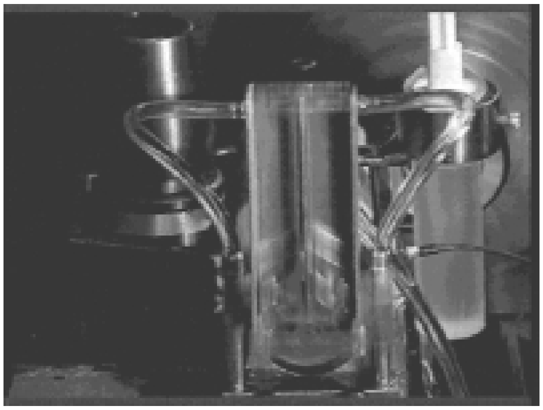

Fig. 5. Oscillating pendulum in the upper equilibrium state

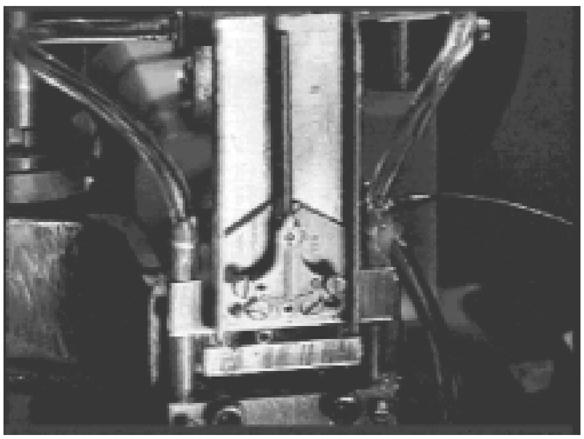

Fig. 6. Pendulum lighted with stroboscope

of technical problems where we can use the benefits of this phenomenon (e.g. the unstable cutting process, or turbulent flows as mentioned in the introduction). The main point is that an unstable oscillation can be stabilized via parametric excitation, that is an oscillation can be eliminated with the help of another oscillation.

Acknowledgement

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References

[1] FARKAS, M. (1994): Periodic Motions, Springer, New York.
[2] HIRSCH, M. W. – SMALE, S. (1974): Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, Berkeley.
[3] SINHA, S. C. – WU, D. H. (1991): An Efficient Computational Scheme for the Analysis of Periodic Systems, *Journal of Sound and Vibration,* Vol. 151, pp. 91–117.