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GENERALIZATION OF CLAPEYRON'S THEOREM OF SOLIDS

Gyula BÉDA

Department of Applied Mechanics
Technical University of Budapest
H–1521 Budapest, Hungary
E-mail: beda@mm.bme.hu

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In memory of Professor Endre Reuss (1900–1968)

Abstract

The mathematical form of Clapeyron's theorem is usually written to solids, which are balanced and suffer small deformations. When the solid moves this mathematical expression should be modified. After an integration with respect to time t the modified Clapeyron's theorem results in the generalization of Clapeyron's theorem.

Keywords: work and power of internal and external forces, moving body, finite deformation.

1. Introduction

Clapeyron's theorem is considered to be a possible basis of the principles of virtual displacements and virtual forces. The theorem is drawn up for solids when they undergo small deformation and they are in state of equilibrium. When the solid moves and its deformation is finite we should modify the theorem. The expedient form of it is given if we modify it from form of work to form of power. We should integrate the form of power according to time t to obtain the original Clapeyron's theorem.

The original Clapeyron's theorem is

$$\int\limits_V t^{ij} u_{i;j} \,\mathrm{d}V = \int\limits_V q^i u_i \,\mathrm{d}V + \int\limits_A t^{ij} n_j u_i \,\mathrm{d}A, \tag{1}$$

that is, the work of internal forces is equal to the work of external forces loading the solid [1], [2].

The t^{ij}, u_i, $u_{i;j}$, q^i are stress, displacement, covariant derivative of displacement, body force and V, A are volume and surface of body.

2. The Modified Clapeyron's Theorem

When the solid moves and undergoes finite deformations we should exchange the displacement u_i for the velocity of particle v_i in *Eq.* (1). Performing it we integrate

Eq. (1) according to time t from t_0 to t_1. We use d'Alembert's principle, that is, body force is completed with $-\rho\dot{v}_i$, where ρ and $\dot{v}_i$ are mass density and acceleration. Now the modified Clapeyron's teorem is

$$\int_{t_0}^{t_1}\int_V t^{ij}v_{i;j}\,\mathrm{d}V\,\mathrm{d}t = \int_{t_0}^{t_1}\int_V b^i v_i\,\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_1}\int_A t^{ij}v_i n_j\,\mathrm{d}A\,\mathrm{d}t, \tag{2}$$

where b_i is identical to $q^i - \rho\dot{v}^i$. Overdot means the material time derivative of the velocity. Volume V and surface A depend on time t. The modified Clapeyron's theorem means the same as the original theorem.

3. The General Clapeyron's Theorem

The two integrals cannot be exchanged. If we want to exchange them we should transform the integrals of volume and surface to the reference configuration where body has volume V_0 and surface A_0. V_0 and A_0 are constant. Now the order of the integration can be exchanged. After these *Eq.* (2) will be

$$\int_{V_0}\int_{t_0}^{t_1} t^{ij}v_{i;j}\overline{J}\,\mathrm{d}t\,\mathrm{d}V_0 = \int_{V_0}\int_{t_0}^{t_1} b^i v_i\overline{J}\,\mathrm{d}t\,\mathrm{d}V + \int_{A_0}\int_{t_0}^{t_1} t^{ij}v_i\overline{J}X^K_{,j}\,\mathrm{d}t\,\mathrm{d}A_{ok}, \tag{3}$$

where $\overline{J} = \dfrac{\mathrm{dV}}{\mathrm{d}V_0}$, $X^K_{,j}$ is the inverse deformation gradient and one can write $v_{i;j}$ as $\dot{u}_{i;K}X^K_{,j}$ [3]. Firstly, performing the time integrals and then returning to the current configuration we obtain

$$\begin{aligned}
&\int_V t^{ij}u_{i;j}\,\mathrm{d}V\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\int_V \left(\dot{t}^{ij} - t^{iq}v^j_{;q} + t^{ij}v^p_{;p}\right)u_{i;j}\,\mathrm{d}V\,\mathrm{d}t \\
&= \int_V b^i u_i\,\mathrm{d}V\Big|_{t_0}^{t_1} + \int_A t^{ij}u_i\,\mathrm{d}A_j\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\int_V \left(\dot{b}^i + b^i v^p_{;p}\right)u_i\,\mathrm{d}V\,\mathrm{d}t \\
&\quad - \int_{t_0}^{t_1}\int_A \left(\dot{t}^{ij} - t^{iq}v^j_{;q} + t^{ij}v^p_{;p}\right)u_i\,\mathrm{d}A_j\,\mathrm{d}t.
\end{aligned} \tag{4}$$

Eq. (4) is regarded as the generalization of Clapeyron's theorem of solids.

The first term on the left hand side and the first and the second terms on the right hand side are equal to each other in *Eq.* (4) while *Eq.* (1) is satisfied from time to time. Therefore, the remaining part of *Eq.* (4) is

$$\begin{aligned}&\int_{t_0}^{t_1}\int_V \left(\dot{t}^{ij} - t^{iq} v^j_{;q} + t^{ij} v^p_{;p}\right) u_{i;j}\, \mathrm{d}V\, \mathrm{d}t \\ &\quad = \int_{t_0}^{t_1}\int_V \left(\dot{b}^i + b^i v^p_{;p}\right) u_i\, \mathrm{d}V\, \mathrm{d}t + \int_{t_0}^{t_1}\int_A \left(\dot{t}^{ij} - t^{iq} v^j_{;q} + t^{ij} v^p_{;p}\right) u_i\, \mathrm{d}A_j\, \mathrm{d}t.\end{aligned} \tag{5}$$

It can be regarded as a generalization of Clapeyron's theorem, too.

References

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