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THE COMPOSITE RATIONAL CURVES AND THEIR SMOOTHNESS

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To memory of Prof. Julius Strommer

Abstract

A model for computing the weights of the control vertices of a rational curve with respect to the continuity constraints is presented. The described method generates for one control polygon a family of curves created from many rational curve segments. The join points of the adjoining curve segments lie along some edges of the control polygon and each curve segment can be controlled by different number of vertices. An initial presentation of the (2,2)-rational patch and the conditions of positional and cross-derivative continuity for linking two patches is offered.

Keywords: rational curve, continuity constraints, weight, rational surface.

1. Introduction

Many different models of spline curves have been introduced in CAD and these models provide an interpolation or approximation of the given set of the points (control polygon). In some applications we are asked to suggest a curve that copies the shape of the control polygon very closely. So closely that the curve can touch some edges of the control polygon. For some applications or some users, the popular models of splines do not give full satisfaction as to the shape of the curve with respect to the foregoing demands. In this situation a composite curve created from the single curve segments presents an available tool.

The main characteristic of this model is that the curve as a whole is created from the curve segments (rational), and the common points of the adjoining curve segments lie along some edges of the control polygon and each curve segment can be controlled by different number of control vertices. The continuity constraints (G^0, C^1, G^1, curvature) in the join point depend on the end-user's demands and they are considered to be the degree of freedom offered by this model. Thus, concentration on computing the weights of some control vertices with respect to the continuity constraints is presented.

2. Preliminary Considerations

The control vertices of a Bezier curve are in an ordered sequence and are connected in succession to form a control polygon. Let the vertices $V_0, \dots, V_n$, with coordinates (V_i^x, V_i^y, V_i^z) be the given points of the Euclidean 3-space and w_i, $i = 0 \dots, n$, be their weights. The points $\bar{V}_i[V_i^x, V_i^y, V_i^z, w_i]$, $i = 0, \dots, n$, are defined in the Euclidean 4-space as vertices of the control polygon of the curve which is expressed by the form

$$\bar{R}(t) = \sum_{i=0}^{n} \bar{V}_i B_{n,i}(t) \qquad t \in \langle 0, 1\rangle, \tag{1}$$

where $B_{n,i}(t)$ are Bernstein polynomials and the curve $\bar{R}(t)$ is *an integral Bezier curve*.

Now we define a central projection with the centre $O(0, 0, 0, 0)$ and a hyperplane $\rho : x_4 = 1$. This central projection maps the vertices $\bar{V}_i$, $i = 0, \dots, n$, of the control polygon in 4-space into the vertices V_i, $i = 0, \dots, n$, in the hyperplane ρ, and the image of the curve (1) in the hyperplane ρ has the analytic representation

$$\bar{r}(t) = \sum_{i=0}^{n} V_i \frac{w_i B_{n,i}(t)}{\sum_{i=0}^{n} w_i B_{n,i}(t)} \qquad t \in \langle 0, 1\rangle \tag{2}$$

and the curve $\bar{r}(t)$ is known as *a rational Bezier curve*.

3. Modification of the Blending Functions

With respect to the next considerations, we deal with the control polygon defined by vertices $V_o, \dots, V_{2k}$, $k = 1, 2, \dots, n = 2k$ [HOSCH]. The weight vector $W(w_0, w_1, \dots, w_{2k-1}, w_{2k})$ has the prescribed value of the weights: $w_0 = w_{2k} = 1$, $w_i > 0$, $i = 1, \dots, 2k-1$ [JOSC]. Now we carry out the following substitutions in expression (2)

$$-\, w_i B_{2k,i}(t) = w_i \binom{2k}{i} t^i (1-t)^{2k-i} = \alpha_i t^i (1-t)^{2k-i}, \tag{3}$$

where $\alpha_i = w_i \binom{2k}{i}$.

Using this substitution, the defined weight vector $W(1, w_1, \dots, w_{2k-1}, 1)$ and the identity

$$(1-t)^{2k+1} + t^{2k+1} = (1-t)^{2k} + \sum_{i=1}^{2k-1} (-1)^i t^i (1-t)^{(2k-i)} + t^{2k}$$

the sum of the combinations (3) can be supplied by the function

$$\omega(\alpha, t) = (1-t)^{2k+1} + \sum_{i=1}^{2k-1} (\alpha_i - (-1)^i) t^i (1-t)^{2k-i} + t^{2k+1}.$$

Now the rational curve (2) has the parameter representation

$$\bar{r}(\alpha, t) = \sum_{i=0}^{2k} V_i RB_{2k,i}(\alpha, t), \qquad t \in < 0, 1 >, \tag{4}$$

where $RB_{2k,i}(\alpha, t)$ are the rational functions of the form

$$\begin{aligned} RB_{2k,0}(\alpha, t) &= \frac{(1-t)^{2k+1}}{\omega(\alpha, t)}, \\ RB_{2k,i}(\alpha, t) &= \frac{(\alpha_i - (-1)^i) t^i (1-t)^{2k-i}}{\omega(\alpha, t)} \qquad i = 1, \ldots, 2k-1, \\ RB_{2k,2k}(\alpha, t) &= \frac{t^{2k+1}}{\omega(\alpha, t)}, \end{aligned} \tag{5}$$

and the curve $\bar{r}(\alpha, t)$ is called *a modified rational curve*.

We note that the rational functions $RB_{2k,i}(\alpha, t)$ are not negative over the interval $\langle 0, 1 \rangle$, form a partition of unity and depend on the parameters $\alpha_i = w_i \binom{2k}{i}$, where w_i are the weights of the control vertices.

4. Why Do We Look for Such Curves?

The basic geometric properties of these rational curves in the endpoints of the given control polygon are as follows:

- the curve interpolates the endpoints;
- the first derivative of the curve in the endpoints is

$$\begin{aligned} \bar{r}'(\alpha, 0) &= (\alpha_1 + 1)(V_1 - V_0), \\ \bar{r}'(\alpha, 1) &= (\alpha_{2k-1} + 1)(V_{2k} - V_{2k-1}); \end{aligned}$$

- the curvature of the curve in the endpoints is defined as

$$k(\alpha, 0) = \frac{2(\alpha_2 - 1)|V_1 - V_2||\sin \gamma_1|}{(\alpha_1 + 1)^2 |V_1 - V_0|^2},$$

$$k(\alpha, 1) = \frac{2(\alpha_{2k-2} - 1)|V_{2k-2} - V_{2k-1}||\sin \gamma_{2k-1}|}{(\alpha_{2k-1} + 1)^2 |V_{2k-1} - V_{2k}|^2},$$

$\gamma_i = \angle V_{i-1} V_i V_{i+1}$, $i = 1, 2k-1$, and $\alpha_i = w_i \binom{2k}{i}$ (see *Fig. 1*).

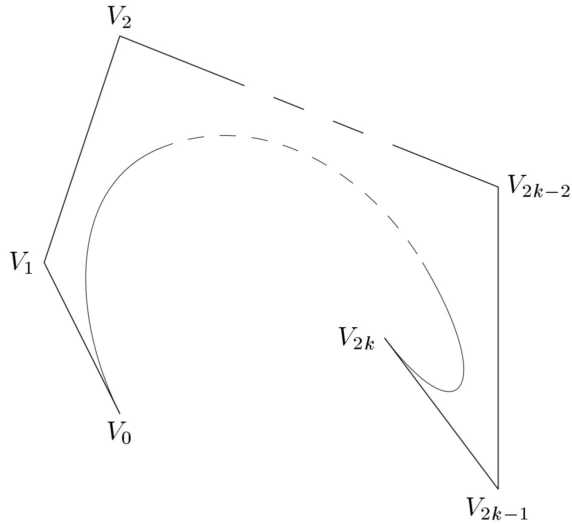

Fig. 1.

These properties of the curve have been applied to 'sew together' two rational curves of this kind. Let $CP[P_i, i = 0, \ldots, 2l, l = 1, 2, \ldots]$ be a control polygon and $\bar{r}^P(\alpha^P, t), t \in \langle 0, 1\rangle$ its curve, and the weight vector has the prescribed form $W^P(1, w_1^P, \ldots, w_{2l-1}^P, 1)$ where $w_i^P = \alpha_i / \binom{2k}{i}$.

This curve $\bar{r}^P(\alpha^P, t)$ is joined with the second one. Its control vertices are called $Q_0, \ldots, Q_{2m}, m = 1, 2, \ldots$ and the weight vector is $W^Q(w_0^Q, \ldots, w_{2m}^Q)$.

The aim is to compute the weights $w_i^Q, i = 0, 1, 2$, of control vertices Q_i according to the demands of continuity:

1. G^0: *geometric continuity*

$$P_{2l} = Q_0 \text{ and the weights } w_{2l}^P = w_0^Q = 1$$

2. C^1: *parametric continuity*
   vertices P_{2l-1}, P_{2l}, Q_1 are collinear, and the equivalence of the first derivatives in the endpoints offers:

$$\frac{P_{2l} - P_{2l-1}}{Q_1 - P_{2l}} = \frac{\alpha_1^Q + 1}{\alpha_{2l-1}^P + 1}$$

   G^1: *geometric continuity*
   to have more flexibility for the composite curve, a coefficient of homothety $k, k > 0$ (later a ratio κ) can be applied. It means

$$|P_{2l} - P_{2l-1}| = k|Q_1 - P_{2l}|$$

and the weight w_1^Q of the control vertex Q_1 is

$$w_1^Q = \frac{k(2l w_{2l-1}^P + 1) - 1}{2m}$$

3. *Equivalence of curvatures*
by equating the values of the curvature in the endpoints, the value of the weight w_2^Q of the control vertex Q_2 is computed

$$w_2^Q = \frac{1}{m(2m-1)}\left[1 + \frac{(l(2l-1)w_{2l-2}^P - 1)|P_{2l-2} - P_{2l-1}||\sin\gamma_{2l-1}^P|}{|Q_1 - Q_2||\sin\gamma_1^Q|}\right].$$

After computing the weights w_0^Q, w_1^Q, w_2^Q of the control vertices Q_0, Q_1, Q_2, the composite curve with respect to the demands of continuity can be constructed.

Now we concentrate on the application of the foregoing curves. We suggest a piecewise curve created from modified rational curve segments of formula (4). The resulting composite curve will mimic a shape of the given control polygon so closely that the curve will touch some edges of the control polygon. The selection of the touching points along the edges of the control polygon will depend on the user's demands, and these points will be the join points of the adjoining curve segments.

Let us have a control polygon $CP[V_i, i = 0, \ldots, n]$. A composite curve of a sequence of $(r+1)$ segments is constructed so that its segments will touch the r-edges $(1 \le r \le n-2)$ of the control polygon $CP[V_i, i = 0, \ldots, n]$ and the adjoining curve segments in the common point will fulfill the conditions for smooth connection.

Now the control polygon is divided into $(r+1)$ subpolygons. With respect to the described method for generating the curve segment and computing the weights, it is important to have odd number of control vertices for each control subpolygon. This problem can be solved by the method – *degree elevation* [FARIN]. Using this method the new control vertices and their weights can be immediately derived:

$$\bar{V}_i' = \frac{w_{i-1}s_i V_{i-1} + w_i(1-s_i)V_i}{w_{i-1}s_i + w_i(1-s_i)},$$

$$w_i' = w_{i-1}s_i + w_i(1-s_i),$$

$i = 0, \ldots, d+1$, $s_i = i/(d+1)$, d – number (odd) of the endpoints of the control subpolygon.

Now having the control subpolygons with odd number of control vertices (it can be different for each control polygon) we can compute their weights for the weight vector according to the demands of continuity applied in the join points. The coefficient of homothety k (G^1 continuity) is computed by a ratio κ of three collinear vertices of two adjoining subpolygons and $k = -\kappa$. The ratio κ is computed for each edge along which a join point of two adjoining curve segments has been chosen. According to the constraints of continuity the weights of the first three control points are computed.

As results, the points on the curve segments are computed and a smooth composite curve copying closely the shape of the original control polygon is constructed. An example:
Fig. 2: two single modified rational curves, the shape is modified by the different weights of the control vertices.
Fig. 3: two composite curves given by the same control polygon, only the join points are different.

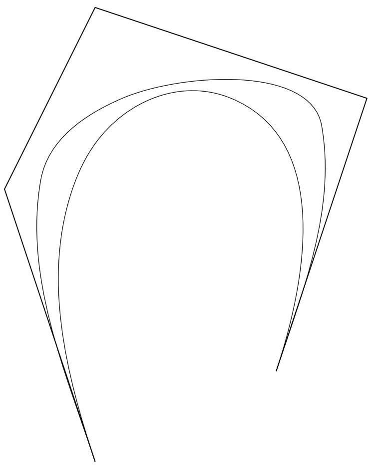

Fig. 2.

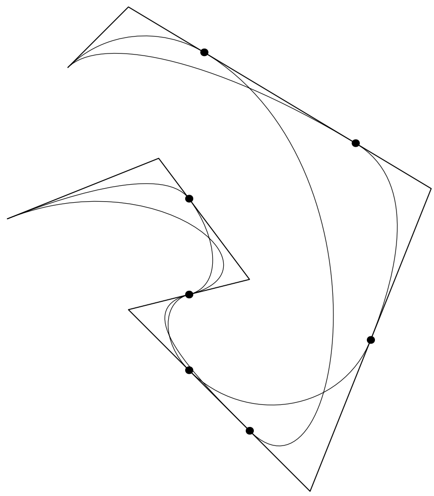

Fig. 3.

5. How to Extend These Results to the Patch?

The extension of the model from curves to patches is straightforward. In the first approach, we start with the parametric representation of the patch

$$\mathbf{r}(\alpha, u; \beta, v) = \sum_{i=0}^{2} \sum_{j=0}^{2} \mathbf{V}_{ij} RB_{2i}(\alpha, u) RB_{2j}(\beta, v),$$

$$u, v \in \langle 0, 1\rangle \times \langle 0, 1\rangle,$$

where V_{ij} are the vertices of the control net (polyhedron) and $RB_{2i}(\alpha, u)$, $RB_{2j}(\beta, v)$ are the rational blending functions of formula (5).

The patch is called *(2,2)- rational patch*. The expression of the patch is similar to the representation of a tensor product patch but the denominator in the rational blending functions causes the difference [FARIN].

Some of the geometric properties of the (2,2)-patch are as follows:

- the patch coincides with four given vertices of the control net;
- the boundaries of the patch are rational curves, and the vertices of their control polygons are the corresponding boundary vertices of the control net;
- the tangent vectors of the boundary curves at the patch corners are in the direction of the corresponding polyhedron edges.

Now the (2,2)-patch is defined in terms of the corner vertices, tangent vectors of the boundary curves at the patch corners. These properties of the (2,2)-patch can be used for linking two patches of this kind. Let the control nets of the patches be defined by the vertices $V_{ij}^{(1)}, V_{ij}^{(2)}, i = 0, 1, 2, j = 0, 1, 2$. The vertices $V_{ij}^{(1)}$ of the first control net are given and the vertices $V_{ij}^{(2)}$ of the second one will be computed. It means that after applying the conditions of

- positional continuity
- cross-derivative

(or tangent continuity in the cross-direction across the common boundary curve of two adjoining patches) the 0-thread (control vertices $V_{i0}^{(2)}, i = 0, 1, 2$) and 1-thread (vertices $V_{i1}^{(2)}, i = 0, 1, 2$) of the second control net are determined [LARR]. Only the control vertices $V_{i2}^{(2)}, i = 0, 1, 2$ of the 2-thread are suggested according to the user's demands.

This method can be extended to create one patch wide but long stripe-surfaces (linking of m (2,2)-patches in one direction) and to control the shape of the surface with two parameters α, β.

6. Conclusion

In this paper, a model of the composite curve has been presented. This model includes many classic properties of rational curves and concentrates on suggesting a family of the smooth composite curves. It works with constraints of continuity applied in the join point of two adjoining curve segments. According to the user's demands on degree of continuity in the join points lying along some edges of the given control polygon, the weights of the control vertices are computed, and one curve from a family of the curves copying the given control polygon is constructed.

The extension of this model to patches is intended in future. More precisely, the following topics will be discussed:

- conditions for mosaic surface patches;
- patches as tensor product of two modified rational curves (generalized tensor product);
- computing the weights of the vertices of the control net with respect to smooth connection across the common boundary curve.

References

[FARIN] FARIN, G. E.: *Curves and Surfaces for Computer Aided Geometric Design*. Academic Press, 3. Edition 1993.

[JOSC] JORDAN, M. C. – SCHINDLER, F.: Curves under Tension, *Computer Aided Geometric Design* **1**, 1984.

[HOSCH] HOSCHEK, J. – LASSER, D.: *Grundlagen der geometrischen Datenverarbeitung*, B.G. Teubner, Stuttgart, 1989.

[LARR] LANG, J. – RASSI, H. – ROESCHEL, O.: GC1-Continuity of Integral Bezier Patches – Another Approach, *Grazer Math. Ber.* Nr.332, 1997.