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SOME PROBLEMS IN WAGNER SPACES WITH VANISHING DOUGLAS TENSOR

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Abstract

In this paper we consider a two dimensional Wagner space of Douglas type with zero curvature scalar, and we give the main scalar function of this space.

Keywords: two-dimensional Finsler spaces, Wagner space, Douglas tensor, projective change.

1. Introduction

In 1943 WAGNER [1] gave a generalization of the concept of BERWALD space by introducing a new connection with the surviving $(h)h$-torsion. Recently HASHIGUCHI [2] established an exact formulation of such a concept based on a theory of Finsler connections developed by M. MATSUMOTO. Throughout the present paper, we shall use the terminology and definitions described in MATSUMOTO's monograph [3].

Definition 1 Let F^n be a Finsler space with a fundamental function $L(x, y)$, $y^i := \dot{x}^i$ and let $g_{ij}(x, y)$ be the fundamental tensor of F^n. There exists a unique Finsler connection $W\Gamma(s) = (F^i_{jk}, N^i_j, C^i_{jk})$ satisfying the following four conditions. This $W\Gamma(s)$ is called the Wagner connection with respect to s_i.

(1) It is h- and v-metrical: $g_{ij||k} = 0$ and $g_{ij}\big\|_k = 0$, where the symbols $\|$ and $\big\|$ mean the covariant derivatives in F^n.

(2) The $(h)h$-torsion tensor $T^i_{jk}(= F^i_{jk} - F^i_{kj})$ is defined by $T^i_{jk} = \delta^i_j s_k - \delta^i_k s_j$, where s_i is a given covariant vector field the components of which are functions of position alone.

(3) The $(v)v$-torsion tensor $S^i_{jk}(= C^i_{jk} - C^i_{kj})$ vanishes.

(4) The deflection tensor $D^i_j(= y^\gamma F^i_{\gamma j} - N^i_j)$ vanishes.

If we denote by $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0j}, C^i_{jk})$, the Cartan connection of F^n, the above C^i_{jk} of $W\Gamma(s)$ are nothing, but those of $C\Gamma$ and the difference $D^i_{jk} = F^i_{jk} - \Gamma^{*i}_{jk}$ are given by

$$D^i_{jk} = L^2(S^i_{jkr} + C^i_{js}C^s_{kr})s^r + (y^i C_{jkr} - y_j C^i_{kr} - y_k C^i_{jr}) + C^i_{jk}s_0 + g_{jk}s^i - \delta^i_k s_j, \quad (1)$$

where $s^i = g^{ij}s_j$ and S^i_{jkr} is the v-curvature tensor of $C\Gamma$. Throughout this work the subscript 0 stands for contraction by y^j.

Definition 2 If a Wagner connection $W\Gamma(s)$ of a Finsler space F^n is linear, namely the connection coefficients F^i_{jk} of $W\Gamma(s)$ are functions of position x^i alone, F^n is called a Wagner space with respect to the vector field $s_i(x)$.

We have many interesting results concerned with Wagner spaces.

Theorem 1 (HASHIGUCHI [2]) *A Finsler space is a Wagner space with respect to a vector field $s_i(x)$, if and only if the C-tensor C_{hij} satisfies $C_{hij\|k} = 0$ (Covariant derivative with respect to $W\Gamma(s)$).*

M. MATSUMOTO [4] found all the Wagner spaces of dimension two.

In the present paper we shall restrict our consideration to two-dimensional Wagner spaces so we drop some words about Berwald frame.

The special and useful Berwald frame was introduced and developed by BERWALD [5], [8]. We study two dimensional Finsler space and define a local field of orthonormal frame (l, m) called the Berwald frame. We give a normalized supporting element $l^i = \frac{y^i}{L}$ and another further on let be given the fundamental tensor with the following equation:

$$g_{ij} = l_i l_j + m_i m_j.$$

In the present paper we give an example for Wagner–Douglas space in the two dimensional case, and we determine its main scalar. We will use the following notions:

$$\begin{aligned} l^i &= \frac{y^i}{L}, \\ l_i &= \dot\partial_i L, \text{ where } \dot\partial_i = \frac{\partial}{\partial y^i}, \\ h_{ij} &= L\dot\partial_i\dot\partial_j L, \\ g_{ij} &= l_i l_j + h_{ij}. \end{aligned}$$

Since the angular metric tensor h_{ij} has the matrix (h_{ij}) of rank $n-1$, we can define the vector $m = (m_1, m_2)$ by $h_{ij} = \varepsilon m_i m_j$, $\varepsilon = \pm 1$. (The sign ε is called the signature of F^2.)

Then we get

$$g_{ij} = l_i l_j + \varepsilon m_i m_j,$$
$$\det(g_{ij}) = \varepsilon (l_1 m_2 - l_2 m_1)^2.$$

The C-tensor ($C_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k (\frac{L^2}{4})$) has no components in the direction l^i ($C_{ijk} y^i = 0$). The tensor C_{ijk} is written using the frame (l, m) in the following formula:

$$LC_{ijk} = I m_i m_j m_k.$$

The scalar field I is called the main scalar of F^2.

Now we define the covariant differentiations in F^2.
Denote by ; , . and |, $\big|$ the covariant differentiations with respect to the Berwald connection $B\Gamma(G^i_{jk}, G^i_j, 0)$ and with respect to the Cartan connection $C\Gamma(\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$ respectively. Then for scalar field $S(x, y)$ we get the following derivations:

$$S_{;i} = S_{|i} = \partial_i S - (\dot{\partial}_r S) G^r_i,$$
$$S_{.i} = S\big|_i = \dot{\partial}_i S.$$

We write $S_{|i}$ and $LS\big|_i$ in the frame (l, m) as follows:

$$S_{|i} = S_{,1} l_i + S_{,2} m_i,$$
$$LS\big|_i = S_{;1} l_i + S_{;2} m_i.$$

$(S_{,1}, S_{,2})$ and $(S_{;1}, S_{;2})$ are called the h- and the v-scalar derivatives of S.

The commutation formulæ for scalar derivatives are written in the form

$$\begin{cases} (1)\ S_{,1,2} - S_{,2,1} = -RS_{;2}, \\ (2)\ S_{,1;2} - S_{;2,1} = S_{,2}, \\ (3)\ S_{,2;2} - S_{;2,2} = -\varepsilon(S_{,1} + IS_{.2} + I_{,1} S_{;2}), \end{cases} \tag{2}$$

where R is called curvature scalar of F^2.

Finally the Bianchi identities for an F^2 reduce to the following identity:

$$I_{,1,1} + RI + \varepsilon R_{;2} = 0. \tag{3}$$

As it is well known, the Berwald connection coefficients G^i_j, G^i_{jk} can be derived from the function G^i, namely $G^i_j = G^i_{.j}$ and $G^i_{jk} = G^i_{.j.k}$, where

$$G^i = \frac{1}{4} g^{is} \left(y^r \left(\frac{\partial L^2_{.s}}{\partial x^r} \right) - \frac{\partial L^2}{\partial x^s} \right).$$

Let us consider two Finsler spaces: $F^n(M^n, L)$ and $\hat{F}^n(M^n, \hat{L})$ on a common underlying manifold M^n.

Definition 3 The change $L(x, y) \to \hat{L}(x, y)$ of metrics is called projective and F^n is projective to $\hat{F}^n$ if any geodesic of F^n is a geodesic of $\hat{F}^n$ as a point set and vice versa.

Definition 4 A Finsler space is called projectively flat, if it has a covering by coordinate neighborhoods in which it is projective to a locally Minkowski space.

From $G_{hjk} = G^i_{.h.j.k}$ we get a projective invariant $\mathcal{D}^i_{hjk}$ called the Douglas tensor [5], [8], [7]:

$$\mathcal{D}^i_{hjk} = G^i_{hjk} = \frac{1}{n+1}(y^i G_{hj.k} + \delta^i_h G_{jk} + \delta^i_j G_{kh} + \delta^i_k G_{hj}),$$

where $G_{hj} = G^r_{hjr}$ and $G_{hj.k} = \dot{\partial}_k G_{hj}$. In particular the $\mathcal{D}^i_{hjk}$ of a two dimensional Finsler space F^2 can be written in the form [10]:

$$3L\mathcal{D}^i_{hjk} = -(6I_{,1} + \varepsilon I_{2;2} + 2I I_2) m_h l^i m_j m_k, \tag{4}$$

where

$$I_2 = I_{,1;2} + I_{,2}.$$

Thus there arises an interesting question: Can we give the main scalar of a Wagner space of Douglas type in an exact formula?

2. The Main Scalar of a Special Wagner Space of Douglas Type

Further on we use the following results:

Theorem [8] If F^n is an n-dimensional, projectively flat Finsler space, then the $v(h)$-torsion tensor $\mathcal{W}^h_{ij}$ and the projective $v(h)$-curvature tensor $\mathcal{D}^h_{ijk}$ are zero identically.

It is a well known result that the Douglas tensor and Weyl tensors vanish identically in a projectively flat Finsler space [6].

MATSUMOTO proved [7]: $\mathcal{W}^h_{ij} = 0$ and $\mathcal{D}^h_{ijk} = 0$ imply:

(1) $H_{ijk} = 0$ and $K_{ij} = 0$, if the dimension is higher than two.
(2) The two dimensional case: $H_{ijk} = 0$, where $H_{ijk} = H_{i.j.k} + G_{jk;i}$, $H_i = L(3Rl_i + R_{.2}m_i)$, and

$$G_{jk} = I_2 m_j m_k / L. \tag{5}$$

Let us consider a two dimensional Wagner space with vanishing Douglas tensor with the following assumption $R = 0$.

From $R = 0$ we get $H_i = 0$.

In the case of $n = 2$, from the theorems above we have $G_{jk;i} = 0$.

Now, using the following formulas:

$$
\begin{aligned}
m_{i,j} &= l^i_{;j} m_i + l^i m_{i;j} = 0,\\
l^i m_i &= 0,\\
l^i_{ij} &= 0,
\end{aligned}
$$

we finally obtain by the help of (5):

$$G_{jk;i} = I_{2;i}\frac{m_j m_k}{L}, \text{ that is } I_{2;2} = 0.$$

We obtain from (4):

$$3I_{,1} + I I_2 = 0. \tag{6}$$

If we use MATSUMOTO's conditions [4] for the Wagner spaces,

$$
\begin{aligned}
I_{,1} &= I_{;2} s_2,\\
I_{,2} &= -I_{;2}(s_1 + I s_2),\\
(s_1)_{;2} - s_2 &= 0,\\
(s_2)_{;2} + s_1 + I s_2 &= 0,\\
(s_1)_{;1} &= 0,\\
(s_2)_{;1} &= 0,
\end{aligned}
$$

then we get the following equations:

$$I_{;2,1} = I_{;2;2} s_2, \tag{7}$$

$$I_2 = I_{;2;2} s_2 - 2I_{;2}(s_1 + I s_2). \tag{8}$$

Using by the Bianchi identity the *Eqs.* (7) and (8) we have:

$$I_{;2;2} s_2 = -\frac{I_{;2} s_{2,1}}{s_2},\ s_2 \neq 0. \tag{9}$$

(If $s_2 = 0$, then a two dimensional Wagner space is a Landsberg space. We know that an F^2 Finsler space is a Landsberg if and only if $I_{,1} = 0$ [4].)

Substituting (9) into (8), then from (8) follows

$$I_2 = -\frac{I_{;2} s_{2,1}}{s_2} - 2I_{;2}(s_1 + I s_2). \tag{10}$$

Now (6) leads to

$$I_{;2}(3s_2 - 2I(s_1 + I s_2) - \frac{I s_{2,1}}{s_2}) = 0. \tag{11}$$

If $I_{;2} = 0$, then a Wagner space is a Berwald space [4], [9].

If we put $I_{;2} \neq 0$, then (11) implies

$$3s_2 - 2I(s_1 + Is_2) - \frac{Is_{2,1}}{s_2} = 0. \tag{12}$$

This is a quadratic equation. Consequently the main scalar is written as

$$I = \frac{-(2s_1 + \frac{s_{2,1}}{s_2}) \pm \sqrt{4s_1^2 + \frac{4s_1 s_{2,1}}{s_2} + \frac{s_{2,1}^2}{s_2^2} + 24s_2^2}}{4s_2}. \tag{13}$$

In general for the main scalar in (13), $I_{;2} \neq 0$. Thus we have the following:

Theorem 2 *The main scalar of a two dimensional Wagner space of Douglas type with the assumption $R = 0$ can be given in the formula (13).*

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