

THE GENERALIZED PRINCIPLE OF VIRTUAL WORK

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Abstract

When the virtual work is considered as a time integral of virtual power, a generalized form of the virtual work principle is obtained. The Euler-Lagrange equation of it gives an equation for the divergence of the Truesdell rate of stress. The equation of motion on the stress rate field is one of the results of this paper.

Keywords: virtual power, equation of motion, Truesdell rate.

1. Introduction

The generalization of virtual work principle emerges in the investigation of third order wave or for example in supervision of finite element method. The problem will be raised in case of third order wave.

The investigation of the third order wave necessitates the knowledge of the dynamic compatibility equation. This equation rises from the first equation of motion in case of the acceleration wave. Now it needs the time derivative of the first equation of motion. The material time derivative isn't simple in the current configuration. Using the principle of virtual power, namely the principle of virtual work also for finite deformation, the derivative will be obvious and indisputable. We assume that the integral of the virtual power with respect to time is the virtual work. Hence, from the principle of virtual work the time derivative of the first equation of motion can be obtained and then the dynamical compatibility equation can be calculated. The time derivative of the first equation of motion will be called the equation of motion on the stress rate field. Many authors have dealt with this problem in the case when the body was in equilibrium [8, 9, 10].

2. The Principle of Virtual Work

In continuum mechanics the principle of virtual power is:

$$\int_V \dot{t}^{kl} v_{k;l}^* dV = \int_V q^k v_k^* dV + \int_{A_p} \tilde{p}^k v_k^* dA \quad , \quad (1)$$

where t^{kl} , v_k^* , $v_{k;l}^*$ and q^k denote the Cauchy stress, the virtual velocity, the virtual velocity gradient and the difference between the body force and the force of inertia in domain V and \tilde{p}^k is the surface force on boundary surface A_p . A is $A_v + A_p$, on A_v the velocity \tilde{v}^k is known.

The stress tensor in (1) satisfies the second Cauchy equation of motion, that is, $t^{kl} = t^{lk}$.

Assume as a starting point that the integral of the power for a given period $[t_1, t_2]$ means the work during this period. Thus, (1) integrated with respect to time t gives

$$\int_{t_1}^{t_2} \int_V t^{kl} v_{k;l}^* dV dt = \int_{t_1}^{t_2} \int_V q^k v_k^* dV dt + \int_{t_1}^{t_2} \int_A \tilde{p}^k v_k^* dA dt. \quad (2)$$

As it can be seen, the virtual deformation rate v_{kl}^* on the left hand side of the equation has been replaced by virtual velocity gradient $v_{k;l}^*$. This replacement leaves the product $t^{kl} v_{kl}^*$ unaltered since $t^{kl} = t^{lk}$. The material time derivative of the deformation gradient is

$$\dot{x}_{;K}^k = v_{;p}^k x_{;K}^p$$

and from this,

$$v_{;p}^k = \dot{x}_{;K}^k X_{;p}^K \quad (3)$$

With the displacement vector \mathbf{u} used and the derivative thereof with respect to time, then to X^K , written in indexed form are

$$v^k = \dot{u}^k \quad \text{and} \quad \dot{x}_{;K}^k = \dot{u}_{;q}^k x_{;K}^q \equiv \dot{u}_{;K}^k,$$

respectively.

Thus, (3) becomes

$$v_{;p}^k = \dot{u}_{;K}^k X_{;p}^K = \dot{u}_{;p}^k. \quad (4)$$

With the volume integral on the left side of (2) transformed to the initial configuration, the integrals with respect to time and over volume V_0 can be interchanged:

$$\int_{V_0} \int_{t_1}^{t_2} t_k^l \dot{u}_{;K}^{*k} X_{;l}^K \bar{J} dt dV_0 = \int_{V_0} \int_{t_1}^{t_2} \bar{J} q^k \dot{u}_k^* dt dV_0 + \int_{A_p^0} \int_{t_1}^{t_2} \bar{J} t^{kl} \dot{u}_k^* X_{;l}^K dt dA_K^0, \quad (5)$$

where $\bar{J} = \frac{dV}{dV_0}$.

Consider now the integrals with respect to time, one after the other:

$$\int_{t_1}^{t_2} \bar{J} t_k^l X_{;l}^K \dot{u}_{;K}^{*k} dt = \int_{t_1}^{t_2} [(\bar{J} t_k^l X_{;l}^K u_{;K}^{*k}) - (\bar{J} t_k^l X_{;l}^K) u_{;K}^{*k}] dt.$$

The first integral can be calculated from time t_1 to t_2 on the right side, that is,

$$\int_{t_1}^{t_2} \bar{J} t_k^l X_{;l}^K \dot{u}_{;K}^{*k} dt = (\bar{J} t_k^l u_{;l}^{*k})_{t_1}^{t_2} - \int_{t_1}^{t_2} \bar{J} (t_k^p v_{;q}^q + \dot{t}_k^p - t_k^l v_{;l}^p) X_{;p}^K u_{;K}^{*k} dt. \quad (6a)$$

After similar transformations, the first integral with respect to time on the right side of (5) is as follows:

$$\int_{t_1}^{t_2} \bar{J} \dot{q}^k \dot{u}_k^* dt = (\bar{J} q^k u_k^*)_{t_1}^{t_2} - \int_{t_1}^{t_2} \bar{J} (\dot{q}^k + v_{;s}^s q^k) u_k^* dt \quad (6b)$$

Here also, virtual displacement $u_k^* \neq 0$ at time t_1 and t_2 in (6b).

Finally, after transformation of the second integral on the right side of (5),

$$\int_{t_1}^{t_2} \bar{J} t^{kl} X_{,l}^K \dot{u}_k^* dt = (\bar{J} t^{kl} X_{,l}^K u_k^*)_{t_1}^{t_2} - \int_{t_1}^{t_2} \bar{J} (t^{kp} v_{;s}^s + \dot{t}^{kp} - t^{kl} v_{;l}^p) X_{,p}^K u_k^* dt. \quad (6c)$$

With Eqs. (6a), (6b) and (6c) substituted into (5) and after proper rearrangement, the principle of virtual work is [1].

$$\begin{aligned} \int_{t_1}^{t_2} \int_V (\dot{t}^{kp} - t^{kq} v_{;q}^p + t^{kp} v_{;s}^s) u_{k;p}^* dV dt &= \int_V [(t_{;l}^{lk} + q^k) u_k^*]_{t_1}^{t_2} dV \\ &+ \int_{A_p} [(\tilde{p}^k - t^{kl} n_l) u_k^*]_{t_1}^{t_2} dA \\ &+ \int_{t_1}^{t_2} \int_V (\dot{q}^k + q^k v_{;s}^s) u_k^* dV dt \\ &- \int_{t_1}^{t_2} \int_{A_p} (\dot{t}^{kp} + t^{kp} v_{;s}^s - t^{kl} v_{;l}^p) n_p u_k^* dAdt, \end{aligned} \quad (7)$$

therefore

$t_{k;l}^l + q_k = 0$ is the first Cauchy equation of motion

and

$\tilde{p}^k \equiv t^{kp} n_p$ dynamic boundary condition on A_p

can be obtained from the first and second terms of the right hand side of the Eq. (7).

3. The Equation of Motion on the Stress Rate Field

The Eq. (7) refers to continue and its any part. Otherwise, on the basis of all that has been mentioned above, the Euler-Lagrange equation given below is obtained after the suitable mathematical transformation:

$$\overset{\circ}{t}_{;p}^{kp} + t^{qp} v_{;q}^k + \check{q}^k = 0 \quad (8)$$

supposing that the Cauchy equations of motion are satisfied. Here $\overset{\circ}{t}^{kp}$ denotes the Truesdell rate of Cauchy's stress tensor, that is,

$$\overset{\circ}{t}^{kp} \equiv \dot{t}^{kp} - t^{kq} v_{;q}^p - t^{qp} v_{;q}^k + t^{kp} v_{;s}^s$$

and

$$\check{q}^k = \dot{q}^k + q^k v_{;s}^s - q^s v_{;s}^k$$

or when \bar{q}^k is the body force density
and

$$q^k \equiv \bar{q}^k - \rho \dot{v}^k \quad \text{then} \quad \check{q}^k = \dot{\bar{q}}^k + \bar{q}^k v_{;s}^s - \bar{q}^s v_{;s}^k - \rho(\ddot{v}^k - \dot{v}^s v_{;s}^k),$$

where ρ is the mass density and it satisfies the continuity equation.

The (8) is the equation of motion on the stress rate field (7), [8, 9, 10]. The boundary condition on A_p is $t^{kp} n_p + t^{qp} n_p v_{;q}^k = 0$.

4. Conclusion

The principle of virtual work is extended to continue, which performs finite deformation. The deformation depends on time, too. The equation of motion for stress rate is derived from the generalized principle.

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