periodica polytechnica

Mechanical Engineering
56/1 (2012) 55–61

web: http://www.pp.bme.hu/me

RESEARCH ARTICLE

Boundary-value problem for the stability of asymmetrically built and loaded multi-layered circular sandwich type plates.

Lajos Pomázi
Received 2012-04-30

Abstract

The present report gives the formulation of the mechanical/mathematical models of the stability of asymmetrically built and loaded circular multi-layered sandwich-type plates with (constructionally) orthotropic hard and transversally isotropic soft layers. Using tensor formalism the corresponding basic formulas are given and the governing equations and natural boundary conditions are derived

Keywords

stability · sandwich · circular

Acknowledgement

The connecting research was supported by the National Research Foundation OTKA N^0 T016041 which support is gratefully acknowledged.

Lajos Pomázi

Department of Applied Mechanics, BME, H-1111 Budapest, Műegyetem rkp. 5, Hungary

1 Formulation of the Problem

In the theory of structures with combined materials the sandwich-type ones plays special role because of their good and predictable physical characteristics. Theories of classical three and regularly multi-layered sandwich plates and shells are done well in the literature. From the point of view of practical applications irregularly built multi-layered plate and shell structures are also very important. Our investigations are dealing with such structures, closely the aim of them are the comparative study of the analytical, experimental and numerical solution of the stability for asymmetrically built and loaded multi-sandwich rectangular plates with (constructionally) orthotropic hard and transversally isotropic soft layers, using the common suppositions in the theory of sandwich and sandwich-type structures (Hoff (1950), Bolotin (1965), Sun et al. (1968)).

Our foregoing research dealt with the stability of regularly multi-layered [5] and asymmetrically built and loaded three- and multi-layered rectangular sandwich-type (consisting of alternating each other hard and soft layers) plates [4],[6]. In the latter case - continuing the proceedings - to the formulation of the stability task it was supposed, that

- material of all layers are elastic and orthotropic (transversally isotropic) or in the case of constructive anisotropy the layout of the hard layers permits to use the "effective stiffness theory", i.e. by "smoothing" (energetically) the stiffness characteristics of the reinforced layers the stiffness of equivalent flat layers can be determined;
- the "hard" layers constitute elastic plates obeying the Kirchhoff-Love laws;
- in the "soft" layers the antiplane shear stresses and in the "transversally soft" layers moreover the antiplane normal stresses are also characteristics and all these are constant across the soft layer thickness and proportional to the corresponding strains. These mean, that the displacement fields of soft layers are linear and determinate by the adjacent hard layers (Mindlin-type layers)[1].
- according to the loading of the plate it is supposed, that the hard layers are loaded with a constant normal and tangential

to the boundary membrane forces only, but this forces could be different - in particular case zero - for different layers and directions.

In the following for the investigations of the stability of circular sandwich plate all these suppositions are saved.

The main differences in the tasks of the stability of the rectangular and circular sandwich plates are in the geometry. Many of authors at formulation of different tasks of circular plates for deriving the corresponding strain components simply are going out from their form in the Descartes coordinates and use transformation rules between the coordinate systems [2], [3]. This method can be work well in simply tasks, but in the case of more complex structures some members of the basic equations can be missed. To avoid this problem in the given paper by formulation of the mathematical- mechanical model of the stability of multi-layered circular sandwich-type plate from the beginning the correct tensor formalism in the system of polar coordinates was used. The corresponding formulas - well known in the thin elastic plate theory - are given in the Appendix 1.

2 Mathematical – Mechanical Modeling

2.1 Initial position

Let us investigate the stability of the asymmetrically built multi-layered circular sandwich-type plate of radius r_0 with n (constructionally) orthotropic hard and $n-1$ transversally isotropic soft layers of thickness h_λ and s_λ ($\lambda = 1, 2$) correspondingly, loaded on the boundary of the hard layers by normal $\left(N_\lambda^{11}\right)$ and may be tangential $\left(N_\lambda^{12}\right)$ distributed in-plane edge forces.

Let join to the plate a polar coordinate system such a way, that the basic [r,θ] plane of the system coincide with the middle surface of the first hard layer and in normal direction to this plane the hard layers let have labels $\lambda = 1, 2, ..., n$, where $\Delta\lambda = 1$. (In case of symmetrically to the middle plane built plate there is more advantageous to join the coordinate system to the middle surface of the plate and to use a labeling $\lambda = \pm1, \pm2, \pm3, ..., \pm m$, if n is even and $\lambda = 0, \pm2, \pm4, ..., \pm m$, if n is odd, where in both case $m = n-1$ and $\Delta\lambda = 2$.) The λ-th soft layer takes place between the hard layers with labels λ and $\lambda + \Delta\lambda$ (Fig. 1).

Our former investigations for the stability of regularly multi-layered (Pomázi (1974 and 1992)) and asymmetrically built and loaded three-layered plates with constructionally orthotropic hard layers (Pomázi (1980, 1985 and 1990)) give the basic methods, formulas and governing equations of the problem.

2.2 Displacement fields

The mechanical-mathematical model of a multi-layered plate with n hard layers is based on the displacement field of the plate, which could be described by 3n displacement functions (as Lagrangean coordinates) in the coordinate directions $u_\alpha(x,y)$, $v_\alpha(x,y)$, $w_\alpha(x,y)$ of the points belonging to the basic surfaces of the hard layers. The displacements in the hard layers are determined by the Kirchhoff-Love law; in the soft layers - based

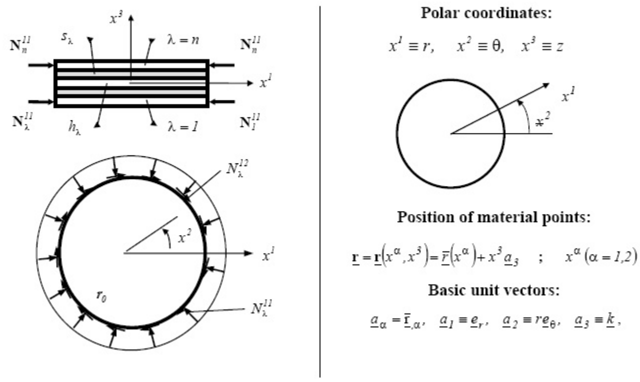


Fig. 1. Multilayered circular sandwich-type plate and it's basic data

on the suppositions - they are linear functions of local normal coordinates and can be expressed by the displacements of adjacent hard layers. So, using this linearity of displacement functions and the corresponding constitutive equations the strain and stress fields of the hard and soft layers can be obtained (Pomázi (1980 and 1992)).

Going out from the aspects of modeling used in the connected previous works [1], [11] the strain and stress field of the plate in this case will be based also on the displacement fields of the points belonging to the middle surface of the hard layers, consequently the model will be continuous by the in-plane coordinates and discrete in normal direction of the plate, expressed by the labels λ of the layers.

Using the common designations of the mathematical and physical quantities and the basic assumptions for the displacement – strain fields mentioned in point **1.** we have the displacements (Fig. 2):

For hard layers:

$$\bar{\boldsymbol{u}}^\lambda = \begin{bmatrix} \bar{u}_\alpha^\lambda \overset{def}{=} u_\alpha^\lambda + x^3\theta_\alpha^\lambda \\ \bar{u}_3^\lambda \overset{def}{=} w^\lambda ; \qquad (\alpha = 1, 2) \end{bmatrix}$$

$$\boldsymbol{u}^\lambda = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^\lambda = \begin{bmatrix} u \\ rv \\ w \end{bmatrix}^\lambda = \begin{bmatrix} u_\lambda \\ rv_\lambda \\ w_\lambda \end{bmatrix}$$

where u_k ($k = \alpha, 3$ *and* $\alpha = 1, 2$) are the vector coordinates and $u_{x^k} = (u,\ v,\ w)$the physical ones. Relation of these coordinates is given by the well known formula: $u_k = H_k u_{x^k}$ where H_k are the Lame's coefficients ($H_1 = H_3 = 1, \quad H_2 = r$).

For soft layers:

$$\tilde{\boldsymbol{u}}^\lambda = \begin{bmatrix} \tilde{u} \\ r\tilde{v} \\ \tilde{w} \end{bmatrix}^\lambda = \begin{bmatrix} u_C \\ rv_C \\ w_C \end{bmatrix}^\lambda + \frac{\xi}{s_\lambda} \begin{bmatrix} u_D - u_C \\ r(v_D - v_C) \\ w_D - w_C \end{bmatrix}$$

where:

$$u_A = u_{\lambda-\Delta\lambda} - \frac{h_{\lambda-\Delta\lambda}}{2} w_{\lambda-\Delta\lambda,1} \, ;$$
$$u_B = u_\lambda + \frac{h_\lambda}{2} w_{\lambda,1} \, ;$$
$$u_C = u_\lambda - \frac{h_\lambda}{2} w_{\lambda,1} \, ;$$
$$u_D = u_{\lambda+\Delta\lambda} + \frac{h_{\lambda+\Delta\lambda}}{2} w_{\lambda+\Delta\lambda,1}$$

analogously:

$$v_A = ..., \quad v_B = ..., \quad v_C = ..., \quad v_D = ...,$$

and:

$$w_A = w_{\lambda-\Delta\lambda}; w_B = w_C = w_\lambda; w_D = w_{\lambda+\Delta\lambda}.$$

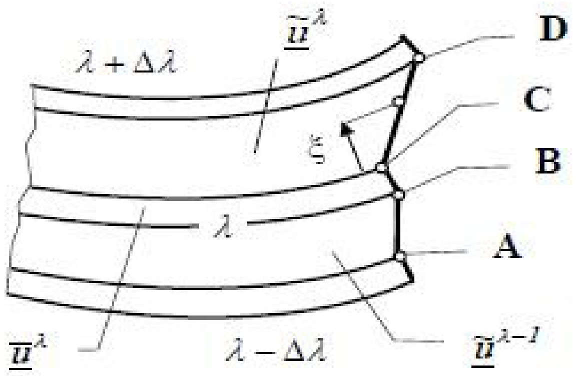


Fig. 2. The displacement field of the layers

2.3 Strain fields:

For hard layers –because of it's plane-shell-type – the strain field can be describe by the covariant derivatives on the middle surface of the displacement field (see Appendix 1.) and so – in general – the strain and curvature tensors are:

$$\bar{\varepsilon}_{\alpha\beta} = \frac{1}{2}(u_{\alpha\|\beta} + u_{\beta\|\alpha}); \quad \kappa_{\alpha\beta} = \theta_{\alpha\|\beta},$$

where by the Kirchhoff - Love law: $\varepsilon_{\alpha 3} = 0$, and so the rotation of the layer's normal:

$$\theta_\alpha = -w,_\alpha - b_\alpha^\beta u_\delta = -w,_\alpha$$

Consequently in the given case the strain field of the hard layers (missing the labels of λ) can be expressed by the following vector/physical coordinates of strain and curvature tensor:

$$\begin{aligned} \varepsilon_{11} &= \varepsilon_r, & \varepsilon_r &= u,_r \, , \\ \varepsilon_{12} &= r\varepsilon_{r\theta}, & \varepsilon_{r\theta} &= \tfrac{1}{2}(\tfrac{u,_\theta}{r} + v,_r - \tfrac{v}{r}), \\ \varepsilon_{22} &= r^2\varepsilon_{\theta\theta}, & \varepsilon_{\theta\theta} &= \tfrac{v,_\theta}{r} + \tfrac{u}{r}, \end{aligned}$$

$$\begin{aligned} & & \kappa_{\theta\theta} &= -w,_{rr} \, , \\ \kappa_{11} &= \kappa_r, & & \\ \kappa_{12} &= r\kappa_{r\theta}, & \kappa_{r\theta} &= -\left(\tfrac{w,_\theta}{r}\right)_{,r} \\ \kappa_{22} &= r^2\kappa_{\theta\theta}, & \kappa_{\theta\theta} &= -\left(\tfrac{w,_{\theta\theta}}{r^2} + \tfrac{w,_r}{r}\right). \end{aligned}$$

For soft layers we have to take into consideration, that displacements of the λ-th hard layer – $u_\lambda,\ v_\lambda,\ w_\lambda$ playing roll later at using variational theorem – will appear in the strains of adjacent λ-th and $\lambda - \Delta\lambda$-th soft layers, therefore the strains for both of soft layers is describing as follows:

$$\begin{aligned} &\tilde{\gamma}_{13}^{\lambda} = \tilde{\gamma}_{rz}^{\lambda} = \tfrac{1}{s_\lambda}\left[u_{\lambda+\Delta\lambda} - u_\lambda + \left(c'_\lambda w_\lambda + c''_\lambda w_{\lambda+\Delta\lambda}\right)_{\|1}\right], \\ &\tilde{\gamma}_{23}^{\lambda} = r\tilde{\gamma}_{\theta z}^{\lambda} = \tfrac{r}{s_\lambda}\left[v_{\lambda+\Delta\lambda} - v_\lambda + \left(c'_\lambda w_\lambda + c''_\lambda w_{\lambda+\Delta\lambda}\right)_{\|2}\right] \\ &\tilde{\varepsilon}_{33}^{\lambda} = \tilde{\varepsilon}_{z}^{\lambda} = \tfrac{1}{s_\lambda}\left(w_{\lambda+\Delta\lambda} - w_\lambda\right), \\ &\tilde{\gamma}_{13}^{\lambda-\Delta\lambda} = \tilde{\gamma}_{rz}^{\lambda-\Delta\lambda} = \\ &\tfrac{1}{s_{\lambda-\Delta\lambda}}\left[u_\lambda - u_{\lambda-\Delta\lambda} + \left(c'_{\lambda-\Delta\lambda} w_{\lambda-\Delta\lambda} + c''_{\lambda-\Delta\lambda} w_\lambda\right)_{\|1}\right] \\ &\tilde{\gamma}_{23}^{\lambda-\Delta\lambda} = r\tilde{\gamma}_{\theta z}^{\lambda-\Delta\lambda} = \\ &\tfrac{r}{s_{\lambda-\Delta\lambda}}\left[v_\lambda - v_{\lambda-\Delta\lambda} + \left(c'_{\lambda-\Delta\lambda} w_{\lambda-\Delta\lambda} + c''_{\lambda-\Delta\lambda} w_\lambda\right)_{\|2}\right] \\ &\tilde{\varepsilon}_{33}^{\lambda-\Delta\lambda} = \tilde{\varepsilon}_{z}^{\lambda-\Delta\lambda} = \tfrac{1}{s_{\lambda-\Delta\lambda}}\left(w_\lambda - w_{\lambda-\Delta\lambda}\right) \end{aligned}$$

Parameters

$$\begin{aligned} c'_\lambda &= 1/2\,(h_\lambda + s_\lambda)\,; \\ c''_\lambda &= 1/2\,(h_{\lambda+\Delta\lambda} + s_\lambda)\,; \\ c'_{\lambda-\Delta\lambda} &= 1/2\,(h_{\lambda-\Delta\lambda} + s_{\lambda-\Delta\lambda})\,; \\ c''_{\lambda-\Delta\lambda} &= 1/2\,(h_\lambda + s_{\lambda-\Delta\lambda}) \end{aligned}$$

2.4 Constitutive laws:

Taking into account that by the basic suppositions material of the hard layers are orthotropic and it is transversally isotropic for the soft layers, for all the layers the Hooke's law is valid: $\boldsymbol{\sigma} = \boldsymbol{B} \cdot \boldsymbol{\varepsilon}$, and

For the hard layers:

$$b_{11} = \frac{E_1}{\bar{\nu}}, \quad b_{12} = \frac{\nu_{12}E_1}{\bar{\nu}}, \quad b_{22} = \frac{E_2}{\bar{\nu}}, \quad b_{66} = G_{12},$$
$$\bar{\nu} = 1 - \nu_{12}\nu_{21}, \nu_{ik}E_k = \nu_{ki}E_i, (b_{ik} = b_{ki})$$

For the transversally soft layers:

$$b_{33} = E' \frac{1-\nu}{1-\nu-2\nu_0\nu'}, \quad b_{44} = b_{55} = G, \quad ,$$

and the Hooke's law:

$$\tilde{\tau}_{rz} = G\tilde{\gamma}_{rz} \, ,$$
$$\tilde{\tau}_{\theta z} = G\,\tilde{\gamma}_{\theta z} \, ,$$
$$\tilde{\sigma}_z = b_{33}, \tilde{\varepsilon}_z,$$

2.5 Generalized constitutive laws:

a) For hard layers:

Hard layers of sandwich-type plates very often are stiffened by stringers or by bending common thin plates. In such "constructionally anisotropic" cases, if on the half-wave length of the solution –functions, characterizing the strain/stress field of the layer there are more stiffeners, then it is possible to formulate the problem for a layer with constant thickness by determining the equivalent stiffness characteristics due to use the effective stiffness theory [11]. The σ and τ stresses in the "generalized constitutive law" usually are coupled except if the stiffened layer's material is isotropic and the layer itself has symmetric characteristics, when the model of the layer will be orthotropic plane-shell.

The strain/stress field of such an equivalent orthotropic hard layer with constant h_λ thickness - on the basic of the Kirchhoff-Love law - can be determined by the $\boldsymbol{\varepsilon}$ strain and $\boldsymbol{\kappa}$ curvature vectors and by the $\boldsymbol{N}$ internal force and $\boldsymbol{M}$bending/torsional moments vectors, getting by integration of the stresses and their static moments on the layer's thickness. Taking ς for local coordinate from the middle plane of the hard layer for these vectors we have:

$$\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon} + \varsigma\boldsymbol{\kappa} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma \end{bmatrix} + \varsigma \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ 2\chi \end{bmatrix},$$

$$\int \boldsymbol{\sigma} d\varsigma = \boldsymbol{N} = \begin{bmatrix} N^{11} \\ N^{22} \\ N^{12} \end{bmatrix},$$

$$\int \boldsymbol{\sigma}\varsigma d\varsigma = \boldsymbol{M} = \begin{bmatrix} \mathrm{M}^{11} \\ \mathrm{M}^{22} \\ \mathrm{H} \end{bmatrix}.$$

The torsion: $H = 2D_{66}.\chi$.

Taking into account the Hooke's law for the stress/train relation by point **2.4** after integration we get the constitutive law for the internal forces and moments in form:

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{K} \\ \boldsymbol{K} & \boldsymbol{D} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}$$

where $\boldsymbol{C}$, $\boldsymbol{K}$, $\boldsymbol{D}$ are the stiffness matrices of the hard layers having forms:

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} & 0 \\ \mathrm{K}_{21} & \mathrm{K}_{22} & 0 \\ 0 & 0 & \mathrm{K}_{66} \end{bmatrix},$$

$$\boldsymbol{D} = \begin{bmatrix} \mathrm{D}_{11} & \mathrm{D}_{12} & 0 \\ \mathrm{D}_{21} & \mathrm{D}_{22} & 0 \\ 0 & 0 & 2\mathrm{D}_{66} \end{bmatrix}.$$

Components of this matrices due to determination of the internal forces/moments are integrals on the thickness of the hard layers of the b_{ik} parameters of the constitutive law of the material of the layers:

$$C_{ik} = \int b_{ik} d\xi, \quad K_{ik} = \int b_{ik}\xi d\xi, D_{ik} = \int b_{ik}\xi^2 d\xi.$$

Matrix $\boldsymbol{K}$express the coupling effect between the stretching and bending of the hard layers, which effect is usual for the (constructionally) orthotropic layer [4],[6].

b) For transversally soft layers:

On the basis of suppositions for the transversally soft layers of constant thickness s, that the strain/stress state of these layers are constant by the normal to the layer, the generalized constitutive law for these layers will be:

$$\tilde{\boldsymbol{N}} = \begin{bmatrix} \tilde{T}^{13} \\ \tilde{T}^{23} \\ \tilde{N}^{33} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{13} & 0 & 0 \\ 0 & \tilde{C}_{23} & 0 \\ 0 & 0 & \tilde{C}_{33} \end{bmatrix} \cdot \begin{bmatrix} \tilde{\gamma}_{13} \\ \tilde{\gamma}_{23} \\ \tilde{\varepsilon}_{33} \end{bmatrix}, \quad = \tilde{\boldsymbol{C}} \cdot \tilde{\boldsymbol{\varepsilon}},$$

where stiffnesses of the layers: $\tilde{C}_{13} = \tilde{C}_{23} = s^2 B, \quad \tilde{C}_{33} = s^2 R.$

Here $B = \frac{\tilde{G}}{s}, \quad R = \frac{\tilde{E}_z}{s}$, where $\tilde{G}$ and $\tilde{E}_z$ are the average of the shear and Young moduli in normal direction of the layer:

$$\tilde{G} = sB = \frac{s}{\int_0^s \frac{1}{G} d\xi}, \tilde{E}_3 = sR = \frac{s}{\int_0^s \frac{1}{b_{33}} d\xi},$$

Physical coordinates :

$$\tilde{T}_\lambda^{13} = \tilde{T}_{rz}^\lambda = s_\lambda^2 B_\lambda \tilde{\gamma}_{rz}^\lambda$$
$$r\tilde{T}_\lambda^{23} = \tilde{T}_{\theta z}^\lambda = s_\lambda^2 B_\lambda \tilde{\gamma}_{\theta z}^\lambda,$$
$$\tilde{N}_\lambda^{33} = \tilde{N}_z^\lambda = s_\lambda^2 R_\lambda \varepsilon_z^\lambda$$

2.6 Relations between the physical and tensor coordinates:

$$N_r = N^{11}, N_{r\theta} = rN^{12}, N_\theta = r^2 N^{22}.$$
$$M_r = M^{11}, M_{r\theta} = rM^{12}, M_\theta = r^2 M^{22}$$
$$\tilde{T}_{rz} = \tilde{T}^{13}, \tilde{T}_{\theta z} = r\tilde{T}^{23}, \tilde{N}_z = \tilde{N}^{33}$$

3 Formulation of the Boundary-value Problem

Following the method given in [1], [4] the governing equations and natural boundary conditions of the given task in the frame of our suppositions can be derived by using the Trefftz variational principle for the functional of total potential energy of the sandwich plate. The corresponding formulas are as follows:

3.1 Energy densities and the total potential energy:

Following the formalism of Bolotin (1965) the strain *energy densities* of hard and *transversally soft* layers (missing the label of the hard layers: λ) are respectively :

$$\left.\begin{aligned} dU &= \frac{1}{2}(\sigma^{11}\varepsilon_{11} + \sigma^{22}\varepsilon_{22} + \sigma^{12}\varepsilon_{12}) + \bar{E}\bar{\varepsilon}^2 O(\delta), \\ d\tilde{U} &= \frac{1}{2}(\tilde{\tau}^{13}\tilde{\gamma}_{13} + \tilde{\tau}^{23}\tilde{\gamma}_{23} + \tilde{\sigma}^{33}\tilde{\varepsilon}_{33}) + \bar{E}\bar{\varepsilon}^2 O(\delta) \end{aligned}\right\}$$

$$\delta \sim \left(\frac{H^2}{l^2}, \frac{E}{\bar{E}}\right),$$

where parameter δ is of second order terms is determined by the quadrate of relations of the full thickness (H) to the characteristic half-wave length of the deformation of the plate (l), or by the relation of the characteristic Young moduli of the soft (E)and the hard $\left(\bar{E}\right)$ layers.

Here the term $\tilde{\sigma}^{33}\tilde{\varepsilon}_{33}$ corresponds to the effect of anti-plane tension/compression of the *transversally soft* layer. If this term is negligibly small compared with others, then can be missed and the layer is as usual simple *soft.* More sophisticated analysis of the characters of the layers is given by Bolotin (1965), Pomázi and Moskalenko (1967).

Neglecting the second order terms, the energy densities are:

$$dU = \frac{1}{2}(\sigma^{11}\varepsilon_{11} + \sigma^{22}\varepsilon_{22} + \sigma^{12}\varepsilon_{12}) \ ,$$
$$d\tilde{U} = \frac{1}{2}(\tilde{\tau}^{13}\tilde{\gamma}_{13} + \tilde{\tau}^{23}\tilde{\gamma}_{23} + \tilde{\sigma}^{33}\tilde{\varepsilon}_{33})$$

After integration of these densities by thickness of the layer we get the surface densities in forms:

$$\Pi_\lambda = \int_{-\frac{h}{2}}^{\frac{h}{2}} dU dx^3, \tilde{\Pi}_\lambda = \int_0^s d\tilde{U} dx^3,$$
$$\Pi_\lambda = \frac{1}{2}\left(N^{\alpha\beta}\varepsilon_{\alpha\beta} + M^{\alpha\beta}\kappa_{\alpha\beta}\right)_\lambda = \frac{1}{2}\left(\boldsymbol{\varepsilon}\,\boldsymbol{C}\,\boldsymbol{\varepsilon} + 2\,\boldsymbol{\varepsilon}\,\boldsymbol{K}\boldsymbol{\kappa} + \boldsymbol{\kappa}\,\boldsymbol{D}\,\boldsymbol{\kappa}\right)_\lambda,$$
$$\tilde{\Pi}_\lambda = \frac{1}{2}\left(\tilde{T}^{\alpha 3}\tilde{\gamma}_{\alpha 3} + \tilde{N}^{33}\tilde{\varepsilon}_{33}\right)_\lambda = \frac{1}{2}\left[\tilde{G}\left(\tilde{\gamma}_{13}^2 + \tilde{\gamma}_{23}^2\right) + \tilde{E}_3\tilde{\varepsilon}_{33}^2\right]_\lambda$$

and calculating also the surface potential energy (work) of external forces $\hat{N}^{\alpha\beta}$ by the formula:

$$\Pi_{L\lambda} = \tfrac{1}{2}\left[\hat{N}^{11}\cdot(u_{3;1})^2 + 2\hat{N}^{12}\cdot(u_{3;1})\cdot(u_{3;2}) + \hat{N}^{22}\cdot(u_{3;2})^2\right]_\lambda$$
$$= \tfrac{1}{2}\left(\hat{N}_\lambda^{\alpha\beta} w_{\lambda\|\alpha}\, w_{\lambda\|\beta}\right)$$

after integration of these on the surface of the layers:

$$U_\lambda = \iint_{(A)} \Pi_\lambda\, dA \ ,$$
$$\tilde{U}_\lambda = \iint_{(A)} \tilde{\Pi}_\lambda dA$$
$$L_\lambda = \iint_{(A)} \Pi_{L\lambda} dA$$

we get the total potential energy of the plate as the functional to be minimize.

$$I\left\langle u_\lambda, v_\lambda, w_\lambda\right\rangle = \sum_{\lambda=1}^{n}(U_\lambda - L_\lambda) + \sum_{1}^{n-1}\tilde{U}_\lambda.$$

3.2 Variational principle:

The Trefftz variational principle: $\delta\left(\delta_*^2 U_0\right) = 0$, where $\delta_*^2 U_0 = I$ (i.e. the second special variation of the total energy for the neutral equilibrium state is equal to the total potential energy for the real small displacements at bifurcation) was analyzed and used for derivation of basic equations of stability and vibrations of regularly multilayered plates by BOLOTIN [1] and also by the author [4–6].

In our case the first variation of functional $I\left\langle u_\lambda, v_\lambda, w_\lambda\right\rangle$ by the displacement functions $u_\alpha^\lambda = (u_\lambda, v_\lambda), w_\lambda$ is connected over the λ-th hard and soft layers also with the soft layer of label: $\lambda - 1$, therefore the Trefftz variational principle: $\delta I = 0$ has form:

$$\delta I\left\langle u_\lambda, v_\lambda, w_\lambda\right\rangle = (\delta U_\lambda - \delta L_\lambda) + \left(\delta\tilde{U}_{\lambda-\Delta\lambda} + \delta\tilde{U}_\lambda\right) = 0.$$

Variations of energy expressions in detail have form:

For hard layers (by label's exchanging):

$$\delta U_\lambda = \iint_{(A)} \left[N^{\alpha\beta}\delta u_{\alpha\|\beta} - \alpha\beta\delta w_{\|\alpha\|\beta} dA\right]_\lambda,$$
$$\delta L_\lambda = \iint_{(A)} \hat{N}_\lambda^{\alpha\beta} w_{\lambda\|\beta} \times \delta w_{\lambda\|\alpha}\; dA.(\alpha,\beta = 1,2),$$

which formulas are the same as for the thin plate. In these expressions and also later in expression of $\delta\tilde{U}_{\lambda-\Delta\lambda} + \delta\tilde{U}_\lambda = \delta\tilde{U}_{\lambda-\Delta\lambda}^\lambda$ it is necessary to eliminate the derivatives of the displacement's variations. This is possible as follows for the first member of dU_λ. Taking into account, that from derivatives of product

$$N^{\alpha\beta}\delta u_{\alpha\|\beta} = \left(N^{\alpha\beta}\delta u_\alpha\right)_{\|\beta} - N_{\|\beta}^{\alpha\beta}\delta u_\alpha,$$

the first member of dU_λ:

$$\iint_{(A)} N^{\alpha\beta}\delta u_{\alpha\|\beta}\, dA = \iint_{(A)} \left(N^{\alpha\beta}\delta u_\alpha\right)_{\|\beta} dA - \iint_{(A)} N_{\|\beta}^{\alpha\beta}\delta u_\alpha.$$

For the first integral on the right hand side let use the Gauss - Ostrogradsky theorem: $\iint_{(A)} \boldsymbol{C}\otimes\nabla\, dA = \oint_L \boldsymbol{C}\cdot\boldsymbol{n}\, ds$, which in the given case (being the bracket (C) scalar) have form: $\iint_{(A)} C_{\|\alpha}\, dA = \oint_L C n_\alpha ds$,

and so we have:

$$\iint_{(A)} \mathrm{N}^{\alpha\beta}\delta \mathrm{u}_{\alpha\|\beta}\, \mathrm{dA} = \oint_{(\mathrm{L})} \mathrm{N}^{\alpha\beta}\delta\mathrm{u}_\alpha \mathrm{n}_\beta \mathrm{ds} - \iint_{(\mathrm{A})} \mathrm{N}_{\|\beta}^{\alpha\beta}\delta\mathrm{u}_\alpha \mathrm{dA}.$$

Taking the same steps for the second member of dU_λ we have:

$$\delta \mathrm{U}_\lambda = \oint_{\mathrm{L}} \left[\mathrm{N}^{\alpha\beta}\mathrm{n}_\beta\delta\mathrm{u}_\alpha - \mathrm{M}^{\alpha\beta}\mathrm{n}_\beta\delta\mathrm{w}_{\|\alpha}\right]_\lambda \mathrm{ds} -$$
$$\iint_{(A)} \left[\mathrm{N}_{\|\beta}^{\alpha\beta}\delta\mathrm{u}_\alpha - \mathrm{M}_{\|\beta}^{\alpha\beta}\delta\mathrm{w}_{\|\alpha}\right]_\lambda \mathrm{dA},$$

where because of derivative of the normal displacement's variation $\delta w_{\|\alpha}$ the last member of this expression has to be integrated and it is:

$$\iint_{(A)} \left[M_{\|\beta}^{\alpha\beta}\delta w_{\|\alpha}\right]_\lambda dA = \oint_L \left[M_{\|\beta}^{\alpha\beta} n_\alpha \delta w\right]_\lambda ds -$$
$$\iint_{(A)} \left[\left(M_{\|\beta}^{\alpha\beta}\right)_{\|\alpha} \delta w\right]_\lambda dA$$

Using the Gauss – Ostrogradsky theorem also for the potential of external forces:

$$\delta \mathrm{L}_\lambda = \iint_{(\mathrm{A})} \hat{\mathrm{N}}^{\alpha\beta}_\lambda \mathrm{w}_{\lambda\|\beta} \times \delta \mathrm{w}_{\lambda\|\alpha}$$

$$\mathrm{dA} = \iint_{(\mathrm{A})} \left[\left(\hat{\mathrm{N}}^{\alpha\beta}\mathrm{w}_{\|\beta}\right)_{\|\alpha} \delta \mathrm{w}\right]_\lambda \mathrm{dA} -$$

$$\oint_{\mathrm{L}} \left[\left(\hat{\mathrm{N}}^{\alpha\beta}\mathrm{w}_{\|\beta}\right) \mathrm{n}_\alpha \delta \mathrm{w}\right]^{\mathrm{ds}}_\lambda .$$

For the soft layers (by formulas above):

$$\delta \tilde{U}^\lambda_{\lambda-\Delta\lambda} = \left(\delta \tilde{U}_{\lambda-\Delta\lambda} + \delta \tilde{U}_\lambda\right) = \iint_{A_0} \left(\delta \tilde{\Pi}_{\lambda-\Delta\lambda} + \delta \tilde{\Pi}_\lambda\right) dA$$

where the energy density for the λ-th soft layer - using relations between tensor and physical coordinates by point **2.6 -** is:

$$\tilde{\Pi}_\lambda = \frac{1}{2}\left(\tilde{\mathrm{T}}^\lambda_{\mathrm{rz}}\tilde{\gamma}^\lambda_{\mathrm{rz}} + \tilde{\mathrm{T}}^\lambda_{\theta\mathrm{z}}\tilde{\gamma}^\lambda_{\theta\mathrm{z}} + \tilde{\mathrm{N}}^\lambda_{\mathrm{z}}\tilde{\varepsilon}^\lambda_{\mathrm{z}}\right) =$$

$$\frac{1}{2}\left(\tilde{\mathrm{T}}^{13}_\lambda\tilde{\gamma}^\lambda_{13} + \mathrm{r}\tilde{\mathrm{T}}^{23}_\lambda\tilde{\gamma}^\lambda_{23} + \tilde{\mathrm{N}}^{33}_\lambda\tilde{\varepsilon}^\lambda_{33}\right)$$

and similar expression can be written for the $\lambda - 1$ layer, too.

Taking into account the strain expressions for the soft layers by point **2.3** we have:

$$\delta \tilde{U}^\lambda_{\lambda-\Delta\lambda} =$$

$$\iint_{A_0} \Big[\tilde{T}^{13}_u \delta u_\lambda + \tilde{T}^{23}_u (r\delta v_\lambda) +$$

$$\tilde{N}^{33}_w \delta w_\lambda + \tilde{T}^{13}_w \delta w_{\lambda\|1} + \tilde{T}^{23}_w \delta w_{\lambda\|2}\Big] dA,$$

where coefficients of the displacement's variations (writing up by the physical coordinates of the internal forces using relations of point **2.6**):

$$\tilde{T}^{13}_u = \tilde{T}^{rz}_u = \frac{1}{s_{\lambda-\Delta\lambda}}\tilde{T}^{\lambda-\Delta\lambda}_{rz} - \frac{1}{s_\lambda}\tilde{T}^\lambda_{rz};$$

$$\tilde{T}^{23}_v = \frac{\tilde{T}^{\theta z}_v}{r} = \frac{1}{r}\left(\frac{1}{s_{\lambda-\Delta\lambda}}\tilde{T}^{\lambda-\Delta\lambda}_{\theta z} - \frac{1}{s_\lambda}\tilde{T}^\lambda_{\theta z}\right);$$

$$\tilde{N}^{33}_w = \tilde{N}^z_w = \frac{1}{s_{\lambda-\Delta\lambda}}\tilde{N}^{\lambda-\Delta\lambda}_z - \frac{1}{s_\lambda}\tilde{N}^\lambda_z;$$

$$\tilde{T}^{13}_w = \tilde{T}^{rz}_w = \frac{c''_{\lambda-\Delta\lambda}}{s_{\lambda-\Delta\lambda}}\tilde{T}^{\lambda-\Delta\lambda}_{rz} + \frac{c'_\lambda}{s_\lambda}\tilde{T}^\lambda_{rz};$$

$$\tilde{T}^{23}_w = \frac{\tilde{T}^{\theta z}_w}{r} = \frac{1}{r}\left(\frac{c''_{\lambda-\Delta\lambda}}{s_{\lambda-\Delta\lambda}}\tilde{T}^{\lambda-\Delta\lambda}_{\theta z} + \frac{c'_\lambda}{s_\lambda}\tilde{T}^\lambda_{\theta z}\right).$$

In integral-expression of $\delta \tilde{U}^\lambda_{\lambda-\Delta\lambda}$ the last two member have derivative of the normal displacement's variation $\delta w_{\|\alpha}$ therefore – similar to the method used at the hard layers above – these members after partial integration and using the Gauss-Ostrogradsky theorem:

$$\iint_{A_0} \left[\tilde{T}^{13}_w \delta w_{\lambda\|1} + \tilde{T}^{23}_w \delta w_{\lambda\|2}\right] dA = \iint_{A_0} \left[\tilde{T}^{\alpha 3}_w \delta w_{\lambda\|\alpha}\right] dA =$$

$$\oint_L \left[\tilde{T}^{\alpha 3}_w n_\alpha \delta w\right] ds - \iint_{A_0} \left[\tilde{T}^{\alpha 3\|\alpha}_w \delta w\right]_\lambda dA,$$

where the surface covariant derivatives: $\tilde{T}^{13}_{w\|1} = \tilde{T}^{13}_w{,}_1$ and $\tilde{T}^{23}_{w\|2} = \tilde{T}^{23}_w{,}_2 + \frac{1}{r}\tilde{T}^{13}$.

Returning to the physical coordinates of the internal forces and to the derivatives by the coordinates using equalities $(\ )_{,1} = (\ )_{,r}$, $(\ )_{,2} = (\ )_{,\theta}$, finally we have:

$$\delta \tilde{U}^\lambda_{\lambda-\Delta\lambda} = \iint_{A_0} \Big\{\tilde{\mathrm{T}}^{\mathrm{rz}}_{\mathrm{u}}\delta \mathrm{u}_\lambda + \tilde{\mathrm{T}}^{\theta z}_{\mathrm{v}}\delta \mathrm{v}_\lambda$$

$$+ \left[\tilde{\mathrm{N}}^{\mathrm{z}}_{\mathrm{w}} - \left(\tilde{\mathrm{T}}^{\mathrm{rz}}_{\mathrm{w},\mathrm{r}} + \frac{1}{\mathrm{r}}\tilde{\mathrm{T}}^{\mathrm{rz}}_{\mathrm{w}} + \frac{1}{\mathrm{r}}\tilde{\mathrm{T}}^{\theta\mathrm{z}}_{\mathrm{w},\theta}\right)\right]\delta \mathrm{w}_\lambda\Big\} dA +$$

$$+ \oint_L \left[\tilde{T}^{rz}_w n_r + \tilde{T}^{\theta z}_w n_\theta\right] \delta w_\lambda ds.$$

Using expressions of δU_λ , δL_λ , $\delta \tilde{U}^\lambda_{\lambda-\Delta\lambda}$ for the first variation of functional $\mathrm{I}\langle u_\alpha, w\rangle$ we get:

$$\delta I\langle u_\alpha, w\rangle = \oint_L \Big[N^{\alpha\beta}n_\beta \delta u_\alpha - M^{\alpha\beta}n_\beta \delta w_{\|\alpha}$$

$$+ \left(M^{\alpha\beta}_{\|\beta} \cdot n_\alpha - \hat{N}^{\alpha\beta}w_{\|\beta} \cdot n_\alpha + \tilde{T}^{\alpha 3}_w n_\alpha\right)\delta w\Big]_\lambda ds$$

$$- \iint_{(A)} \Big\{\left[N^{\alpha\beta}_{\|\beta} + \tilde{T}^{\alpha 3}_u\right]\delta u_\alpha + \Big[\left(M^{\alpha\beta}_{\|\beta}\right)_{\|\alpha}$$

$$+ \left(\hat{N}^{\alpha\beta}w_{\|\beta\|\alpha}\right) + \tilde{T}^{\alpha 3}_{w\|\alpha} + \tilde{T}^{33}_w\Big]\delta\, w\Big\}_\lambda \mathrm{d}A = 0$$

According to the hard layers in expression δU_λ the second member of the boundary integral $-\oint_L \left[M^{\alpha\beta}n_\beta \delta w_{\|\alpha}\right]_\lambda ds$ at $\alpha = 2$ in physical coordinates has form:

$$-\oint_L \left[\frac{M_{\theta r}}{r}n_r + \frac{M_\theta}{r^2}n_\theta\right]\delta w_{,\theta}\, ds =$$

$$-\oint_L \left[\left(\frac{M_{\theta r}}{r}n_r + \frac{M_\theta}{r^2}n_\theta\right)\delta w\right]_{,\theta} \mathrm{d}s$$

$$+ \oint_L \left(\frac{M_{\theta r,\theta}}{r}n_r + \frac{M_{\theta,\theta}}{r^2}n_\theta\right)\delta w\, ds,$$

where the first integral is equal to zero ($ds = r d\theta$), the second one is an additional member in the boundary condition at $\delta w \neq 0$.

On the basis of expressions above we can write the boundary value problem for the stability of multi-layered circular sandwich-type plate in physical coordinates as follows:

3.3 Governing equations (for the l-th layer):

1 $N_{r,r} + \frac{1}{r}N_{r\theta,\theta} + \frac{1}{r}(N_r - N_\theta) + \tilde{T}^{rz}_u = 0$,

2 $N_{r\theta,r} + \frac{2}{r}N_{r\theta} + \frac{1}{r}N_{\theta,\theta} + \tilde{T}^{\theta z}_v = 0,$

3 $M_{r,rr} + \frac{2}{r^2}M_{r\theta,theta} + \frac{2}{r}M_{r\theta,r\theta} - \frac{1}{r}M_{\theta,r} + \frac{2}{r}M_{r,r} + \frac{1}{r^2}M_{\theta,\theta\theta} + \tilde{N}^z_w - \left(\tilde{T}^{rz}_{wr} + \frac{1}{r}\tilde{T}^{rz}_w + \frac{1}{r}\tilde{T}^{\theta z,\theta}_w\right) - \hat{q} = 0$

where: $\hat{q} = \hat{N}_r w_{rr} + 2\hat{N}_{r\theta}\left(\frac{w_\theta}{r}\right)_{,r} + \hat{N}_\theta\left(\frac{w_r}{r} + \frac{w_{\theta\theta}}{r^2}\right)$.

This set of governing equations consists of *3n* difference-differential equation, expressing that the mechanical-mathematical model of the problem is continuous by the in-plane coordinates *x,y* and discrete by the coordinate *z*, perpendicular to the plate, expressed by label: l.

3.4 Natural boundary conditions at r_0 = const.:

In equation $\delta I = 0$ the general expression of the boundary integral in physical coordinates has form:

$$\oint_L \Big\{ \Big[N_r n_r + \frac{N_{r\theta}}{r} n_\theta\Big]\delta u + \Big(\frac{N_{r\theta}}{r} n_r + \frac{N_\theta}{r^2} n_\theta\Big) r\,\delta v - \Big[M_r n_r + \frac{M_{r\theta}}{r} n_\theta\Big]\delta w_{,r}$$
$$+ \Big[\Big(M_{r,r} + \frac{2M_{r\theta,\theta}}{r} + \frac{1}{r}(M_r - M_\theta) - \hat N_r w_{,r} - \frac{\hat N_{r\theta}}{r} w_{,\theta} + \tilde T_w^{rz}\Big) n_r$$
$$+ \Big(\frac1r M_{\theta r,r} + \frac2r M_{r\theta} + \frac{2}{r^2} M_{\theta,\theta} - \frac1r \hat N_{\theta r} w_{,r} - \frac{1}{r^2}\hat N_\theta w_{,\theta} + \tilde T_w^{\theta z}\Big) n_\theta\Big]\delta w\Big\}\, \mathrm{d}s = 0,$$

from which on the boundary $\mathbf{r_0}$ **= const.**$(n_r = 1,\quad n_\theta = 0)$ of the λ-th hard layer:

1 $\delta u_\lambda \neq 0 \quad\rightarrow\quad N_r = 0$

2 $\delta v_\lambda \neq 0 \quad\rightarrow\quad N_{r\theta} = 0$,

3 $\delta w_{,r} \neq 0 \quad\rightarrow\quad M_r = 0$

4 $\delta w_\lambda \neq 0 \quad\rightarrow\quad M_{r,r} + \frac1r\left(2M_{r\theta,\theta} + M_r - M_\theta\right) - \hat N_r w_{,r} - \frac1r \hat N_{r\theta} w_{,\theta} + \tilde T_w^{rz} = 0$

Basic equations for the boundary hard layers give special additional boundary conditions, in which all members of expressions $\tilde T_u^{rz},\ \tilde T_v^{\theta z},\ \tilde N_w^{z}$ related to the missed soft layers on the surface of the plate are equal to zero, expressed by the zero material constants (or zero stiffnesses), i.e.:at

$$\lambda = 1 : B_0 = R_0 = 0 \rightarrow \tilde T_{rz}^0 = \tilde T_{\theta z}^0 = \tilde N_z^0 = 0,$$

at

$$\lambda = n : B_n = R_n = 0 \rightarrow \tilde T_{rz}^n = \tilde T_{\theta_{rz}^n} = \tilde N_z^n = 0$$

These conditions should be taken into account in the boundary conditions on the boundary $\mathbf{r_0}$ **= const.** of the layers $\lambda = 1, n$.

References

1 **Bolotin V.**, *Strength and Vibration of Multilayered Plates*, Strength's Calculations **12** (1965).

2 **Hoff N**, *Bending and Buckling of Rectangular placeSandwich Plates* **NCSA TN 2225** (1950).

3 **Pomázi L, Moskalenko VN**, *On the Effect of the Fulfilment's Poisson Ratios for theStability of Multilayered Plates*, Periodica Polytechnica Mech.Eng **11** (1967), no. 1.

4 **Pomázi L, Moskalenko VN**, *About the Stability of Layered Plates*, Építési kutatás-fejlesztés **11** (1974), no. 3–4, 45–54.

5 **Pomázi L**, *Investigations of the Free Vibrations of Multilayered Plates*, Proceedings of the placeI.National Congress of Theoretical and Applied Mechanics (1975).

6 **Pomázi L**, *On Postbuckling Behavior of Regularly Multi-layered Rectangular Elastic Plates*, Acta Technica Scientiarum Hungariae **87** (1978), no. 1–2, 111–120.

7 **Pomázi L**, *Stability of Rectangular placeSandwich Plates with Constructionally Orthotropic Hard placeGivenNameLayers middlenamePart SnI*, Periodica Polytechnica, Mech. Eng **24** (1980), 203–222.

8 **Pomázi L**, *Further Investigations on the Postbuckling Behavior of Regularly Multilayered Rectangular Elastic Plates*, ZAMM 63 (1983), T84-85.

9 **Pomázi L**, *Remarks to the Stability of Asymmetrically Built and Loaded placeSandwich Plates*, Proceedings of the EUROMECH Colloquium N-200 (1985), 253-275.

10 **Pomázi L**, *Stability of Rectangular placeSandwich Plates with Constructionally Orthotropic HardLayers. Part II.*, Periodica Polytechnica, Mech. Eng **34** (1990), no. 1-2, 99 - 111.

11 **Pomázi L**, *On the Stability and Postbuckling Behaviour of Multilayered Sandwich-type Plates.Int.*, J. of Solids and Structures **29** (1992), no. 14-15, 1969-1980.

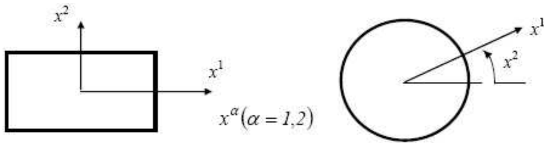
12 **Pomázi L**, *Stability of Asymmetrically Built and Loaded Multilayered Rectangular Sandwich-type Plates (Analytical Solution)*, Publications of the University of placeCityMiskolc, Series C., Mechanical Engineering **45** (1996), 75–286.

13 **Sun CT, Achenbach JD, Herrmann G**, *Continuum Theory for Laminated Medium*, Journal of Applied Mechanics (1968), 467-475.

Appendix 1.

Descartes coordinates: Polar coordinates:

$x^1 \equiv x,\ x^2 \equiv y,\ x^3 \equiv z \quad x^1 \equiv r,\ x^2 \equiv \theta,\ x^3 \equiv z$



Position vector of the material points:

$\boldsymbol{r} = \boldsymbol{r}\left(x^\alpha, x^3\right) = \bar{\boldsymbol{r}}\left(x^\alpha\right) + x^3\boldsymbol{a}_3$

Derivatives and basic unit vectors:

$\boldsymbol{g}_k = \dfrac{\partial \boldsymbol{r}}{\partial x^k} = \boldsymbol{r}_{,k};\quad \boldsymbol{a}_\alpha = \bar{\boldsymbol{r}}_{,\alpha};\quad \boldsymbol{g}_\alpha = \boldsymbol{a}_\alpha\left(x^\beta\right) + x^3\boldsymbol{a}_{3,\alpha};$

$\boldsymbol{g}_3 = \dfrac{\boldsymbol{g}_1 \times \boldsymbol{g}_2}{|\boldsymbol{g}_1 \times \boldsymbol{g}_2|} = \boldsymbol{a}_3 \equiv \boldsymbol{k}$

$\boldsymbol{a}_1 = \boldsymbol{i}\cdot\cos\theta + \boldsymbol{j}\cdot\sin\theta \equiv \boldsymbol{e}_r;$

$\boldsymbol{a}_2 = -\boldsymbol{i}\cdot\sin\theta + \boldsymbol{j}\cdot\cos\theta \equiv r\boldsymbol{e}_\theta$

Christoffel's symbols:

$\boldsymbol{a}_{k,l} = \Gamma_{kl}^m \boldsymbol{a}_m\ (k, l, m = \alpha\, 3\ \alpha = 1, 2)$

$\Gamma_{22}^1 = -x^1 = r; \Gamma_{21}^2 = \Gamma_{12}^2 = \frac{1}{x^1} = \frac1r;$

$\Gamma_{\alpha\beta}^3 \overset{def}{=} b_{\alpha\beta} = 0; \Gamma_{\alpha 3}^\beta = b_\alpha^\beta = 0$

$\boldsymbol{a}_{k,l} = \boldsymbol{a}_{\alpha,\beta} + \boldsymbol{a}_{3,\beta} + \boldsymbol{a}_{3,3} = \boldsymbol{a}_{\alpha,\beta}$ because:

$\boldsymbol{a}_{3,\beta} = -b_\beta^\alpha \boldsymbol{a}_\alpha = 0$ (Weingarten formula),

$\boldsymbol{a}_{3,3} = \Gamma_{33}^\alpha \boldsymbol{a}_\alpha = 0.$

$\boldsymbol{a}_{\alpha,\beta} = \Gamma_{\alpha\beta}^k \boldsymbol{a}_k = \Gamma_{\alpha\beta}^\delta \boldsymbol{a}_\delta + b_{\alpha\beta}\boldsymbol{a}_3 = \Gamma_{\alpha\beta}^\delta \boldsymbol{a}_\delta$ (Gauss formula)

$\boldsymbol{a}_{\alpha,\beta} = \boldsymbol{a}_{\alpha\|\beta} = \boldsymbol{a}_{\alpha,\beta} - \Gamma_{\alpha\beta}^\delta \boldsymbol{a}_\delta = b_{\alpha\beta}\boldsymbol{a}_3 = 0$ (Weingarten's law)

Covariant derivatives on the middle surfaces of the hard layers:

$a_{\alpha\|\beta} = a_{\alpha,\beta} - \Gamma_{\alpha\beta}^\delta a_\delta,\ a_{\|\gamma}^{\alpha\beta} = a_{,\gamma}^{\alpha\beta} + \Gamma_{\gamma\delta}^\alpha a^{\beta\delta} + \Gamma_{\delta\gamma}^\beta a^{\delta\alpha},$

$a_{\gamma\|\varepsilon}^{\alpha\beta} = a_{\gamma,\varepsilon}^{\alpha\beta} + a_\gamma^{\delta\beta}\Gamma_{\delta\varepsilon}^\alpha + a_\gamma^{\alpha\delta}\Gamma_{\delta\varepsilon}^\beta - a_\gamma^{\alpha\beta}\Gamma_{\gamma\varepsilon}^\delta,$

$\Gamma_{22}^1 = -r,\ \Gamma_{21}^2 = \Gamma_{12}^2 = \frac1r.$