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Cycle Bases of Graphs for the Analysis of Frames Structures Using Force Method

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Abstract

The formation of suitable cycle bases corresponding to sparse flexibility matrices for the force method of frame analysis has always been an interesting problem in structural mechanics. These cycle bases are needed for the formation of static bases for efficient force method of structural analysis. Similarly, such bases are required in the mesh analysis of other networks. This paper reviews methods for the cycle basis selection by utilizing different embeddings on higher dimensional topological spaces, and using the ideas and concept from this study, graph theory algorithms are developed for efficient computational algorithms for the formation of subminimal, and minimal cycle bases.

Keywords

force method, flexibility matrix, static basis, cycle basis, minimal cycle basis, subminimal cycle basis

1 Introduction

Cycle bases of graphs have many applications in various fields of engineering. The amount of work in these applications depends on the cycle basis chosen. A basis with shorter cycles reduces the time and storage required for some applications, i.e., it is ideal to select a minimal cycle basis, and for some other applications minimal overlaps of cycles are needed; i.e. optimal cycle bases are preferred. In Section 1, the formation of minimal and subminimal cycle bases is first discussed. Then the possibility of selecting optimal and suboptimal cycle bases is investigated.

Minimal cycle bases were considered first by Stepanets [1] and improved by Zykov [2]. Many practical algorithms for selecting subminimal cycle bases have been developed by Cassell et al. [3], Kaveh [4], Kaveh [5] and Cassell et al. [6]. Similar methods have been presented by Hubicka and Syslø [7]. Kolasińska [8] found a counter example to Hubicka and Syslø's algorithm. A similar conjecture was made by Kaveh [9], for planar graphs; however, a counter example has been given by Kaveh and Roosta [10]. Horton [11] presented a polynomial time algorithm to find minimal cycle bases of graphs using the greedy algorithm. This algorithm has been previously used by Kaveh [4] considering all the simple cycles of the graph, however the algorithm of Horton was more efficient (using smaller number of cycles) which has been further improved by Kaveh and Mokhtar-Zadeh [12].

In the present paper, the merits of the algorithms developed by different authors are discussed; a method is given for selection of minimal cycle bases, and efficient approaches are presented for the generation of subminimal cycle bases. After this introduction in Section 2, the force method is described. In Section 3 embeddings graphs on higher dimensional spaces is described. Section 4 is confined to topological methods for cycle bases selection, followed by Section 5 devoted to graph algorithms for the force method. Examples are given in Section 6. Formation of $\boldsymbol{B}_1$ matrices is described in Section 7, and Section 8 briefly concludes the paper.

2 Force method of structural analysis

In Section 2, the force method of structural analysis is presented using the basic tools namely *equilibrium*, *compatibility* and *load-displacement relationships*. The notations are chosen from those most commonly incorporated in structural mechanics, Kaveh [13].

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2.1 Equilibrium equations

Consider a structure S with $M(S)$ members and $N(S)$ nodes, which is $\gamma(S)$ times statically indeterminate. Select $\gamma(S)$ independent unknown forces as redundants. These unknown forces can be selected from external reactions and/or internal forces of the structure. Denote these redundants by $\boldsymbol{q} = \{q_1, q_2, \ldots, q_{\gamma(S)}\}^t$. Remove the constraints corresponding to redundants, in order to obtain a statically determinate structure, known as the *basic* (*released* or *primary*) structure of S. Obviously a basic structure should be rigid. Consider the joint loads as $\boldsymbol{p} = \{p_1, p_2, \ldots, p_{\gamma(S)}\}^t$, where n is the number of components for the applied loads at nodes.

Now the stress resultant distribution $\boldsymbol{r}$ due to the given load $\boldsymbol{p}$, for a linear analysis by the force method, can be written as

$$\boldsymbol{r} = \boldsymbol{B}_0\boldsymbol{p} + \boldsymbol{B}_1\boldsymbol{q}, \tag{1}$$

where $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are rectangular matrices, each having m rows, and n and $\gamma(S)$ columns, respectively, m being the number of independent components for member forces. $\boldsymbol{B}_0\boldsymbol{p}$ is known as a *particular solution* which satisfies equilibrium with the imposed load and $\boldsymbol{B}_1\boldsymbol{q}$ is a *complementary solution* formed from a maximal set of independent self-equilibrating stress systems (S.E.Ss), known as a *statical basis*.

2.2 Compatibility equations

The compatibility equations in the actual structure will now be derived. Using the load-displacement relationship for each member, and collecting them in the diagonal of the unassembled flexibility matrix $\boldsymbol{F}_m$, one can write member distortions as

$$\boldsymbol{u} = \boldsymbol{F}_m\boldsymbol{r} = \boldsymbol{F}_m\boldsymbol{B}_0\boldsymbol{p} + \boldsymbol{F}_m\boldsymbol{B}_1\boldsymbol{q}\,, \tag{2}$$

in matrix form

$$\left[\boldsymbol{u}\right] = \left[\boldsymbol{F}_m\right]\left[\boldsymbol{B}_0 \quad \boldsymbol{B}_1\right]\begin{bmatrix}\boldsymbol{p}\\ \boldsymbol{q}\end{bmatrix}. \tag{3}$$

From the contragradient principle we have,

$$\left[\boldsymbol{v}\right] = \begin{bmatrix}\boldsymbol{B}_0^t\\ \boldsymbol{B}_1^t\end{bmatrix}\left[\boldsymbol{u}\right]. \tag{4}$$

Combining Eq. (3) and Eq. (4) results in

$$\begin{bmatrix}\boldsymbol{v}_0\\ \boldsymbol{v}_c\end{bmatrix} = \begin{bmatrix}\boldsymbol{B}_0^t\boldsymbol{F}_m\boldsymbol{B}_0 & \boldsymbol{B}_0^t\boldsymbol{F}_m\boldsymbol{B}_1\\ \boldsymbol{B}_1^t\boldsymbol{F}_m\boldsymbol{B}_0 & \boldsymbol{B}_1^t\boldsymbol{F}_m\boldsymbol{B}_1\end{bmatrix}\begin{bmatrix}\boldsymbol{p}\\ \boldsymbol{q}\end{bmatrix}, \tag{5}$$

where $\boldsymbol{v}_0$ contains the displacements corresponding to the components of $\boldsymbol{p}$, and $\boldsymbol{v}_c$ denotes the relative displacements of the cuts for the basic structure. Performing the multiplication and defining the following:

$$\begin{aligned}&\boldsymbol{D}_{00} = \boldsymbol{B}_0^t\boldsymbol{F}_m\boldsymbol{B}_0,\ \ \boldsymbol{D}_{10} = \boldsymbol{B}_0^t\boldsymbol{F}_m\boldsymbol{B}_1,\\ &\boldsymbol{D}_{01} = \boldsymbol{B}_1^t\boldsymbol{F}_m\boldsymbol{B}_0,\ \ \boldsymbol{D}_{11} = \boldsymbol{B}_1^t\boldsymbol{F}_m\boldsymbol{B}_1.\end{aligned} \tag{6}$$

Expansion of Eq. (5) results in

$$\boldsymbol{v}_0 = \boldsymbol{D}_{00}\boldsymbol{p} + \boldsymbol{D}_{10}\boldsymbol{q}, \text{ and } \boldsymbol{v}_c = \boldsymbol{D}_{01}\boldsymbol{p} + \boldsymbol{D}_{11}\boldsymbol{q}. \tag{7}$$

Considering the compatibility conditions as $\boldsymbol{v}_c = 0$, the second part of the Eq. (7) leads to

$$\boldsymbol{q} = -\boldsymbol{D}_{11}^{-1}\boldsymbol{D}_{10}\boldsymbol{p}. \tag{8}$$

Substituting in Eq. (7) yields the displacements

$$\boldsymbol{v}_0 = \left[\boldsymbol{D}_{00} - \boldsymbol{D}_{01}\boldsymbol{D}_{11}^{-1}\boldsymbol{D}_{10}\right]\boldsymbol{p}, \tag{9}$$

and the stress resultants of the structure as

$$\boldsymbol{r} = \left[\boldsymbol{B}_0 - \boldsymbol{B}_1\boldsymbol{D}_{11}^{-1}\boldsymbol{D}_{10}\right]\boldsymbol{p}. \tag{10}$$

The force method of frame analysis requires the formation of suitable static bases corresponding to sparse flexibility matrices. Due to the pattern equivalence of the flexibility matrix $\boldsymbol{D}_{11}$ of a frame and the cycle adjacency matrix $\boldsymbol{D}$ of its graph model, the problem can be converted into the formation of a maximal set of independent cycles, known as a *cycle basis*. In order to have a sparse flexibility matrix for a frame structure whose elements have the least overlaps should be selected (*optimal cycle basis*). The formation of an optimal cycle basis is not possible at present; however, such a basis is quite often among cycle bases of the least length (*minimal cycle bases*). There are various methods for the selection of subminimal cycle basis, some of which will be described in subsequent sections.

Some of the methods available for the force method of structural analysis, and the topological transformations involved in different steps of the force method are illustrated in the following flowchart, Fig. 1.

From Fig. 1 it can be seen how the three well-known topological invariants of graphs, namely *crossing number*, *thickness* and *genus* of the graphs are involved in an efficient analysis of frame structures by the force method. Though these problems are NP-complete, however, fortunately subminimal and suboptimal cycle bases are sufficient for solution of the structural problems and absolute minimal property, or optimality of cycle basis is not a necessary requirement.

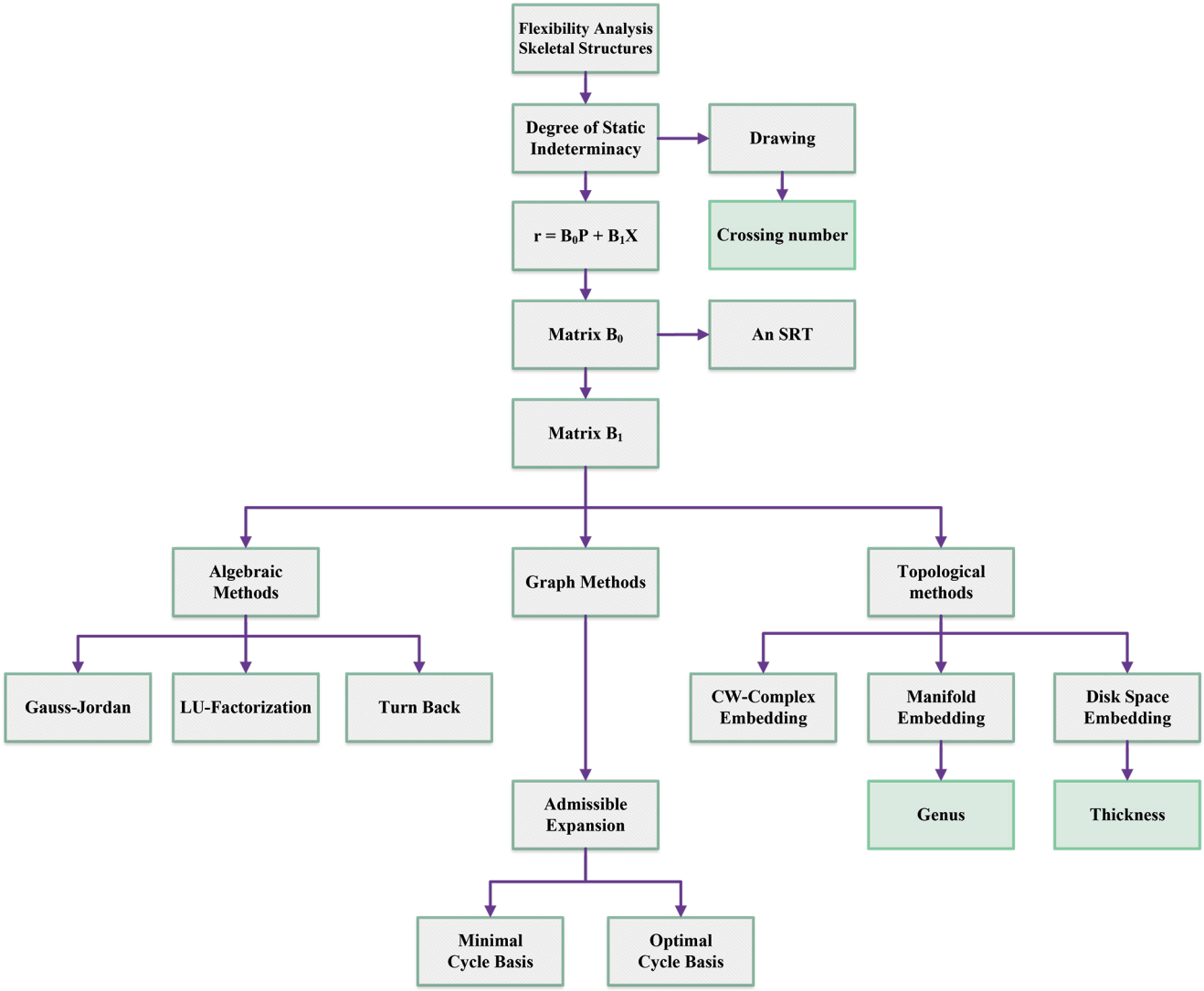


Fig. 1 Flexibility analysis of frame structures and graph invariants

3 Embeddings graphs on higher dimensional spaces

In order to visualize and study certain properties of graphs, it is useful to embed it on higher dimensional spaces such as polyhedrons, manifolds or complexes. Such embeddings highlights the additional properties that can be utilized in the generation of suitable cycle bases corresponding to sparse cycle adjacency matrices, hence leading to sparse flexibility matrices.

In Section 3, some topological methods useful for the formation of cycle bases are presented. These methods are developed by de C. Henderson [14], de C. Henderson and Maunder [15], Maunder [16], and Kaveh [17, 18]. For completeness of the discussion, both topological and graph theory methods are described in this paper. The first class, however, requires some terminology from algebraic and combinatorial topology, and these definitions and concepts are briefly provided. However, for further description, the interested reader may refer to the books of Cooke and Finney [19] and Maunder [20].

3.1 Definitions and concepts from topology and algebraic topology

Manifolds are like curves and surfaces, except that they might have higher dimensions. The dimension of a manifold is the number of independent parameters required to specify one of its points. An n-dimensional manifold is n-dimensional Euclidian space R^n, in which each point is an n-tuple of real numbers. An n-dimensional manifold is an object modeled locally on R^n, i.e., it takes exactly n numbers for specifying a point.

Curves can be considered as manifolds of dimension one. Examples of such manifolds are lines, circles, parabolas or the graph of any continuous function of the form as $y = f(x)$. A point on a curve can be specified by a single real number.

Surfaces are manifolds of dimension 2. The two common examples are planes and spheres. Other familiar surfaces are cylinders, ellipsoids, paraboloids and torus. In these manifold two coordinates are needed to determine a point. For example, on the plane one can use Cartesian or polar coordinates; on the sphere we can use latitude and longitude coordinates, while in torus one may use two angles. The only higher dimensional manifold that can be visualized is Euclidean 3-space. However, it is not hard to construct subsets of higher dimensional spaces that might be called manifolds. As an example, any open subset of R^n is an n-manifold. An n-dimensional manifold "*look like*" R^n locally. To make sense of the notion of "look like", let use consider two subsets of Euclidean spaces $U \subset R^n$ and $V \subset R^n$. These subsets are topologically equivalent (Greek for "same form") if there exists a one-to-one correspondence φ: $U \rightarrow V$ such that both φ and its inverse φ^{-1} are continuous maps. Such a correspondence is called *homeomorphism*. A subset M of some Euclidean space R^k is *locally Euclidean of dimension n* if every point of M has a neighborhood in M that is topologically equivalent to a ball in R^n.

Now the *n-dimensional manifold* (or n-manifold) can be defined as a subset of some Euclidean space R^k that is locally Euclidean of dimension n.

The foundation of topology goes back to the pioneering work of French mathematician Henri Poincaré, as an attempt to classify geometric objects that appear in analysis. Poincaré laid out the main problems of topology and introduced powerful ideas for their solution in a series of papers during 1899-1905. His first paper entitled "*Analysis Situs*" Latin from "analysis of position" is the old name of topology.

Roughly speaking, topology is a branch of mathematics that is concerned with the properties of sets that are unchanged by continuous deformations. More accurately, a topological property is one that is preserved by homeomorphisms. As simple examples of homeomorphism, the following mappings may be considered.

A cube can be continuously deformed into a sphere without tearing. Similarly, a triangle, rectangular and a circle can continuously be deformed to each other without tearing. On the other hand, there is no homeomorphism between a sphere and a torus, since any such a map would have to tear an opening (hole) in the sphere, and thus cannot be continuous. As another example, the deformation of a torus into a coffee cup is illustrated in Fig. 2.

It is rather easy to prove that two manifolds are topologically equivalent. However, it is difficult to prove that two manifolds are not homeomorphic even when the manifolds

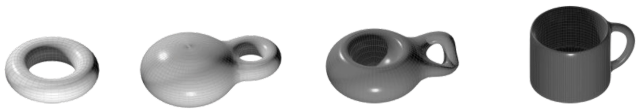


Fig. 2 From a torus to a coffee cup

are simple ones. As an example, sphere and torus are not homeomorphic, however one needs to show that no one, no matter how clever, could find such a map. In order to prove that two given manifolds are not topologically equivalent is to find their topological invariants. These invariants can be numbers or other mathematical objects such groups, matrices, polynomials or vector spaces. Two manifolds with different invariants cannot be homeomorphic.

Obviously geometric properties such as circumference or area are not topological invariants, since they are not generally preserved by homeomorphism. The property which differs for a sphere and a torus is the existence of a hole in the torus. In order to find the number of holes in a manifold the commonly used invariant is the fundamental group of the manifold, which is a group attached to each manifold in such a manner that homeomorphic manifolds have isomorphic groups. The size of the fundamental group is a measure of the number of holes possessed by the manifold.

3.2 Orientable 2-manifolds

If a graph is not planar, it is natural to embed it on other surfaces (2-maifolds). For this purpose, we consider only the orientable compact 2-manifolds. Compact 2-manifolds are surfaces having the following three properties:

1. Each point has a neighborhood homeomorphic to an open disk.
2. Every cover of the surface with open disks has a finite subcovery.
3. Such a surface is orientable if the following additional property holds: a clockwise sense of rotation can be assigned consistently around all points.

Sphere and torus are simplest examples of such surfaces. The plane itself is not qualified, since it is not compact. Less formally an orientable manifold can be considered as a sphere with h handles, denoted by S_h. A double torus as an example is a sphere with two handles, as shown in Fig. 3.

For further definitions the interested reader may refer to Kaveh [20].

It can easily be proved that for a 2-ell embedding of a graph G on an orientable manifold S_h, the polyhedral formula of Euler becomes

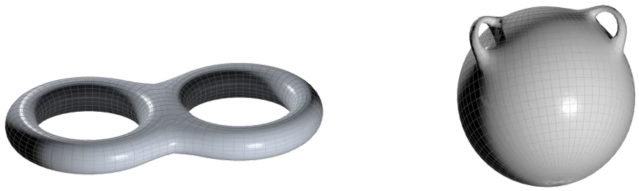


Fig. 3 A double torus and a sphere with two handles

$$R(G)-M(G)+N(G)=2-2h\,, \tag{11}$$

where $R(G)$, $M(G)$ and $N(G)$ are the numbers of regions, edges and node of the embedding.

3.3 Simplicial complexes

An affine map between two vector spaces is a map f: $V \to W$ of the form $f(x) = a(x) + b$, where a is a linear map and $b \in W$. An affine subspace of a vector space is the zero set of some affine map $\{x: a(x) + b = 0\}$. Its dimension is the dimension of the kernel of the linear part of the affine map. We say that $k + 1$ points are in general position if they are not contained in any $(k - 1)$-dimensional affine subspace, or equivalently if $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ are linearly independent.

Given points $v_0, v_1, \ldots, v_k$ in general position of R^n, the simplex spanned by them is the set of all points in R^n of the following form

$$\sum_{i=0}^{k} t_i v_i, \text{ where } 0 \le t_i \le 1 \text{ and } \sum_{i=0}^{k} t_i = 1, \tag{12}$$

with the subspace topology. Each of the points v_i is called a *vertex* of the simplex. The integer k is called the *dimension*, and k-dimensional simplex is often called a *k-simplex*. A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a (filled in) triangle, and a 3-simplex is a solid tetrahedron as shown in Fig. 4.

A Euclidean *simplicial complex* is a collection K of simplices in some Euclidean space R^n satisfying the following conditions:

1. if $\sigma \in K$, then every face of σ is in K.
2. The intersection of any two simplices in K is either empty or in a face of each.
3. Every point in a simplex of K has a neighborhood that intersects at most finitely many simplices of K.

The dimension of K is defined to be the maximum dimension of any simplex in K.

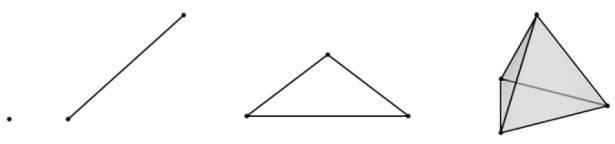


Fig. 4 Simplices of different dimensions

While Fig. 5 (a) shows a 2-complex, Fig. 5 (b) is not a complex since the intersection condition is violated.

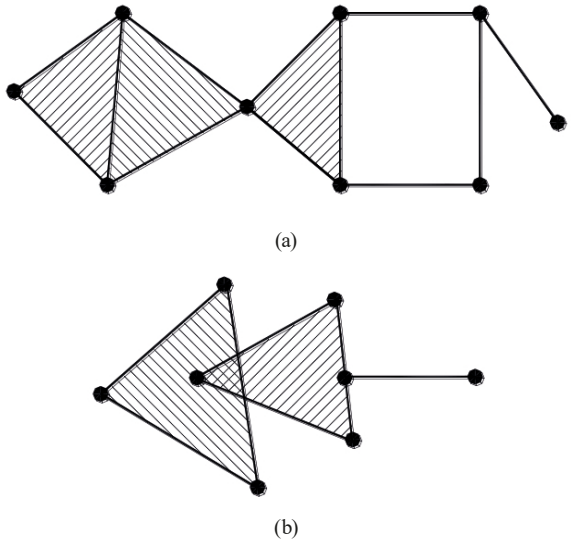
Given a Euclidean complex K, the union of all the simplices in K, with the subspace topology inherited from R^n, is a topological space denoted by $|K|$ and is called the *polyhedron* of K.

Any subset $K' \subset K$ that is itself a simplicial complex is called a *subcomplex* of K. It is clear that the only condition that needs to be checked is that the faces of each simplex in K' are in K. In particular for any nonnegative integer k, the subset $K^{(k)} \subset K$ containing all simplices of dimension less than or equal to k is a subcomplex, called the *k-skeleton* of K. Consider two simplicial complexes K and L. A continuous map f: $|K| \to |L|$ whose restriction to each simplex in K is a simplicial map to a simplex of L is called a simplicial map.

3.4 CW-complexes

Consider a unit interval I as the subspace of R^1, consisting of points x such that $0 \le x \le 1$. The n-cell E^n is the subspace $\{x \in R^n \mid d(x,0) \le 1\}$ and the $(n - 1)$-sphere S^{n-1} is $\{x \in R^N \mid d(x,0) = 1\}$ (E^0 is a single point and S^{-1} is empty).

The notion of simplicial complexes can be simplified and generalized by relaxing all linearity conditions. By building up spaces by successively attaching a finite number of cells to, say, a discrete set of points a CW-complex can be formed. This generalizes the idea of a polyhedron, because the cells are attached by arbitrary continuous maps, and it has the advantage of expressing a given polyhedron as a CW-complex with fewer cells than the original number of simplexes.



(a)

(b)

Fig. 5 (a) A 2-complex and (b) a non-complex

3.5 Collapsible and contractible complexes

Let K be a CW-complex. An n-cell σ of K is said to have a free face τ, if τ is an $(n-1)$-face of σ but is the face of no other n-cell of K. If σ has a free face, it is easy to see that σ is not a proper face of any cell of K, so that K-σ-τ is a subcomplex of K. The process of obtaining K-σ-τ from K is called an *elementary collapse*. If L is a subcomplex of K, then K is said to *collapse* to L, and it is written as $K \rightarrow L$, if L can be obtained from K by a sequence of elementary collapses. As an example, the collapse of a 2-cell into a point is illustrated in Fig. 6, where the free face at each stage is encircled.

The collapse of a 3-simplex into three 2-simplexes and further collapse into 1-simplexes followed by collapse of 1-simplexes into 0-simplex (a single node) is illustrated in Fig. 7. Due to this collapsibility, the boundaries of three 2-simplexes form a cycle basis of S.

Two spaces X and Y are *homotopy-equivalent* (or of the same homotopy type) if there exits maps f: $X \rightarrow Y$ and g: $Y \rightarrow X$, such that gf and fg are identity maps of X and Y, respectively. Subspace A of a topological space X is a *retract* of X if there exists a map r: $A \rightarrow X$, such that $r(a) = a$ for all $a \in A$. A space X, homotopy-equivalent to a point, is called *contractible*. Naturally if a complex is collapsible into a point, then it is also contractible, while the reverse is not always true.

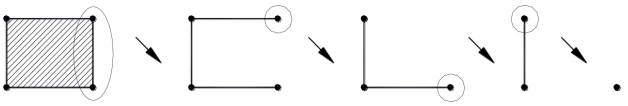


Fig. 6 The collapse of a 2-cell into a point

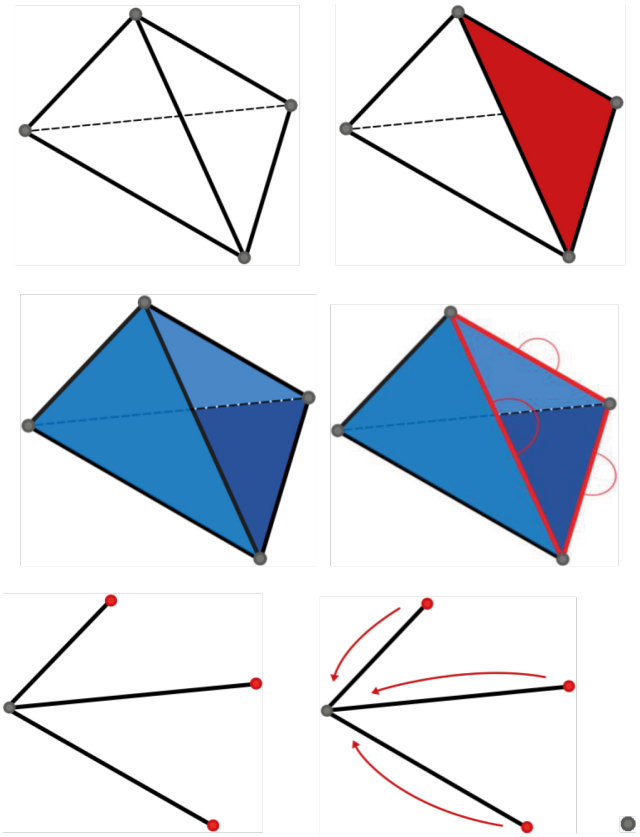


Fig. 7 Collapse of 3-simplex into a single 0-simplex

3.6 Homology group

Given a chain group C, the group of n-boundaries $B_n(C)$ is defined to be the image of ∂: $C_{n+1} \rightarrow C_n$ and the group of n-cycles $Z_n(C)$ is the kernel of ∂: $C_n \rightarrow C_{n-1}$. Clearly, $B_n(C) \subset Z_n(C)$, since $\partial^2 = 0$. The nth homology group $H_n(C)$ is defined to be the quotient group $Z_n(C)/B_n(C)$.

4 Cycle bases selection: topological methods

Cycle selection procedures described in this section use the concept of embedding the geometric realization of S into another polyhedron whose dissection has dimension equal to 2. The idea originates from a planar graph embedded in R^2, in which the cycles bounding finite regions form an efficient basis (mesh basis) for the first cycle group $Z_1(S,R)$.

The object is to extend this approach to embedding S on polyhedrons and manifolds with certain properties. These properties are measured by using the homology groups $H_p(K,R)$ of the underlying complex, which measure, roughly speaking, the number of independent p-dimensional holes of K. In other words, they measure the extent to which K has non-bounding p-cycles.

4.1 A 2-dimensional polyhedron embedding

Let S be the mathematical or geometric realization of S is often denoted by $|S|$. However, for simplicity it will also be denoted by S.

An *embedding* f: $S \rightarrow P$ is a homeomorphism of S into polyhedron P. An embedding is called a *2-cell embedding* if the components of $[P - f(S)]$ are all 2-cells. If the 2-cells are regular, then embedding is called a *regular 2-cell embedding*. Let $f(S)$ be dissected into a 1-complex isomorphic to the dissection of S. Then $f(S)$ and the components of $[P - f(S)]$ from a dissection of P into a 2-dimensional cell complex K.

4.2 Admissible embeddings

The cycles bounding the 2-cells of K are known as *regional cycles*. An *admissible embedding* f of S is one for which the regional cycles from a set of $b_1(S)$ independent cycles from $Z_1(K,R)$. The necessary and sufficient condition for a 2-cell embedding f: $S \rightarrow P$ to be admissible is that the homology groups with real coefficients are trivial. In the

case of 2-cell embedding, this condition holds when first and second Betti number of K are zero. Using concepts from algebraic topology, the admissibility condition can be stated as follows:

The necessary and sufficient condition for f: $S \rightarrow P$ to be admissible is that the corresponding K be acyclic.

A regular complex K is called acyclic if all homology groups of K are trivial. Equivalently, any cycle of $Z_1(K,R)$ bounds in K; i.e. $Z_1(K,R) = B_1(K,R)$.

It is easy show that a contractible complex is acyclic, hence a contractible embedding is admissible. If K is collapsible, then it is obviously contractible. Thus, a collapsible embedding is also admissible.

A graph can be viewed as the 1-skeleton of a 3-complex. An n-cell is called *collapsible* if it can be shrunk into the remainder of its $n - 1$ cells through a free $n - 1$ cell. If a 3-complex can be collapsed into a point, then it is called *collapsible*. A collapsible 3-complex can be used for the formation of a cycle basis of its 1-skeleton. This can be achieved by collapsing all the 3-cells through free 2-cells or 2-cells being freed in subsequent steps [21].

Example 1: The two non-planar graphs K_5 and $K_{3,3}$ are embedded into polyhedrons whose dissections are collapsible. In both cases, we have initially free 1-faces and therefore the embeddings are trivial, Fig. 8.

Example 2: Consider a space graph S as shown in Fig. 9 (a). This graph can be viewed as the 1-skeleton of a 3-complex as depicted in Fig. 9 (b). After collapsing all the 3-cells through three shaded 2-cells (shown in red), the bounding cycles of the remaining 2-complex consisting of 13 four-sided cycles (shown in blue color) form a suitable cycle basis of S which is obviously a minimal cycle basis.

Example 3: Let S be a space graph as shown in Fig. 10 (a). S can be viewed as the 1-skeleton of a 3-complex as illustrated in Fig. 10 (b). After collapsing all the 3-cells through the shaded 2-cells, the bounding cycles of the remaining 2-complex consisting of 81 four sided cycles form a cycle basis of S.

4.3 Modified manifold embedding

Edmond's permutation technique provides a method for a 2-cell embedding of S on an orientable 2-manifold M. The choice of node permutations in this method is made arbitrarily, which may lead to a manifold with an unnecessarily large genus. The *genus* of graph is the minimum number of handles of a sphere on which the graph can be embedded. In order to reduce the genus of $\boldsymbol{M}$, Duke [22] developed a reduction technique for transforming a 2-cell embedding $\boldsymbol{M}$ into $\boldsymbol{M}^*$, where the genus of $\boldsymbol{M}^*$ is one less than that of $\boldsymbol{M}$. This algorithm is very lengthy and

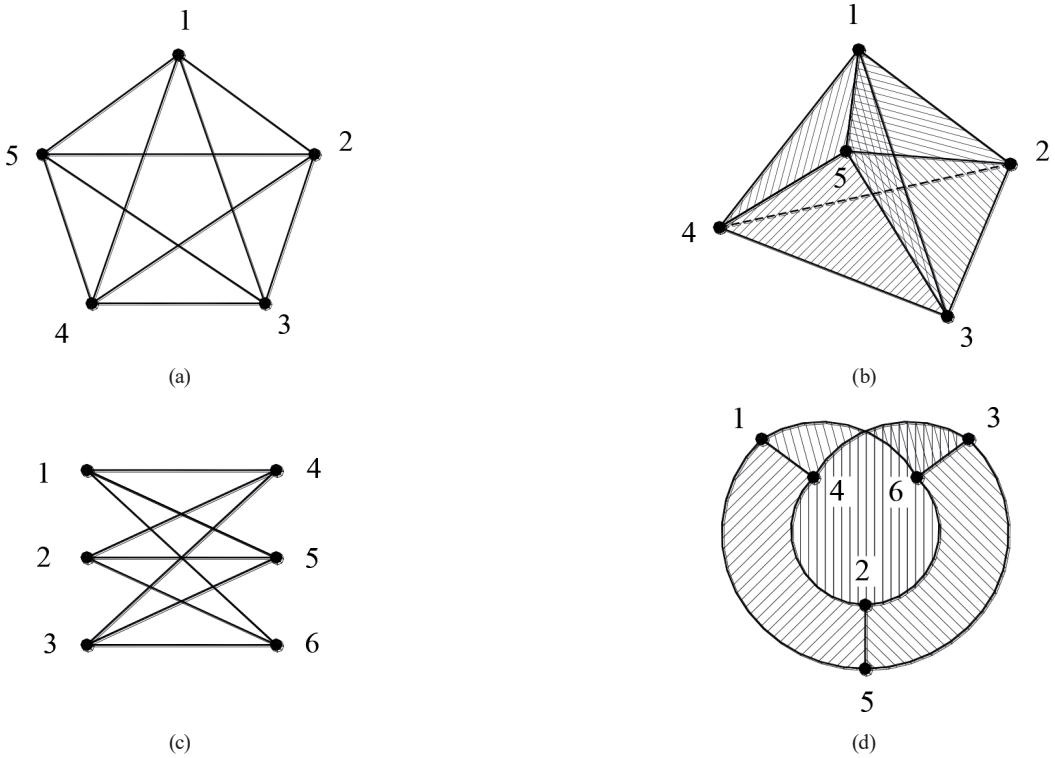
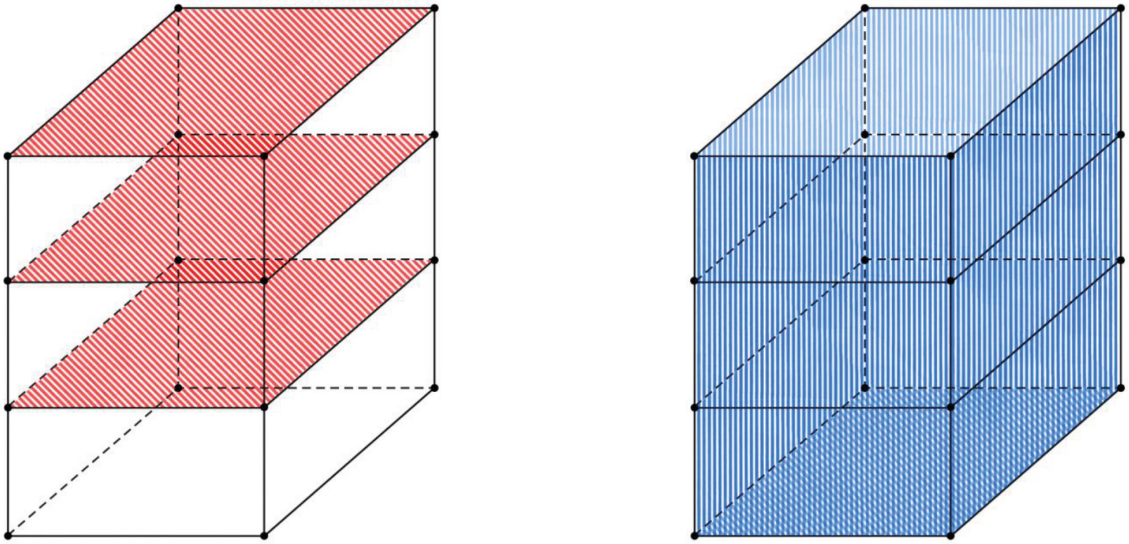
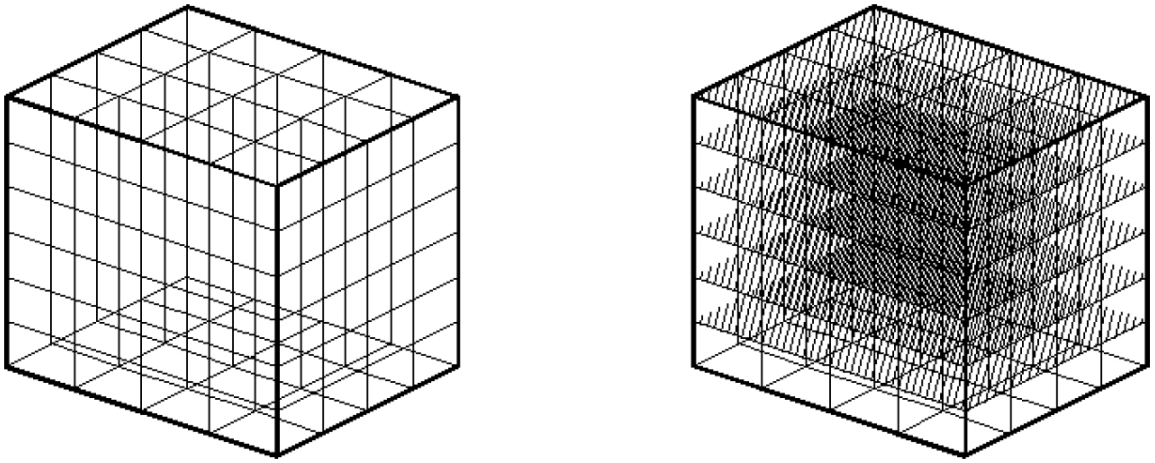


Fig. 8 Embeddings of K_5 and $K_{3,3}$ on polyhedrons whose dissections are collapsible: (a) $S = K_5$, (b) f: $S \rightarrow |K|$, (c) $S = K_{3,3}$, (d) f: $S \rightarrow |K|$



(a) (b)

Fig. 9 A space graph and its collapsible embedding: (a) A space graph *S*, (b) S embedded on a 3-complex



(a) (b)

Fig. 10 A space graph and is collapsible embedding: (a) A space graph *S*, (b) *S* embedded on a 3-complex

impractical. Kaveh [23] developed a practical method in which the node permutations are determined in the process of embedding *S*. However, this algorithm does not always lead to minimal embeddings.

A different approach may also be employed which is based on an intuitive regular embedding of *S* on a manifold. For an embedding f: $S \rightarrow P$ which dissects polyhedron $\boldsymbol{M}$ into 2-complex $\boldsymbol{K}$, $\boldsymbol{M}$ is a manifold if:

1. Each 1-cell of K is incident with exactly two 2-cells.
2. All the 2-cells and 1-cells of K having a particular 0-cell ad a face can be ordered in a sequence so that the consecutive cells are incident.
3. K can be oriented so that $\sum_i \overline{z}_i = 0$, where $\overline{z}_i$ is a regional cycle.

If M is manifold, then $b_2(K) = 1$ and $b_1(K) = 2\Psi$, where Ψ is the genus of $\boldsymbol{M}$, i.e., M is homeomorphic to a sphere with Ψ handles. Thus, f is not admissible, but can be modified by adding 2Ψ appropriate fillings and one perforation of order 2 to make it admissible.

Example 1: Consider a structure *S* with one opening as shown in Fig. 11 (a). This structure being embedded on a torus (a sphere with one handle) shown in Fig. 11 (b). This manifold is modified with the addition of two generators G_1 and G_2 and perforation P as illustrated in Fig. 11 (c). The final embedding of *S* on torus is shown in Fig. 11 (d), where the selected cycles consist of all 4-sides regional cycles together with G_1 and G_2 minus the 4-sided perforation *P*.

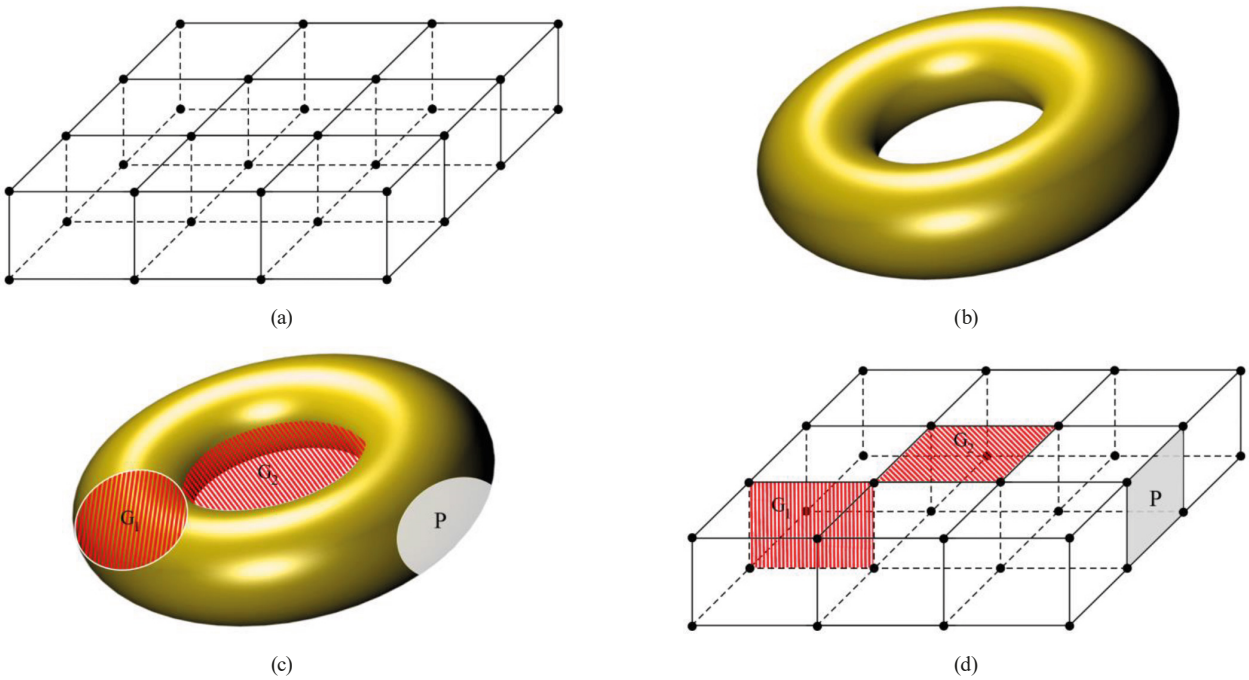


Fig. 11 An admissible embedding of a graph on admissible manifold: (a) a graph *S*, (b) a torus, (c) an admissible embedding, (d) the selected cycle basis

Example 2: Consider a hollow box *S*, as shown in Fig. 12 (a) with two cut-outs which is embedded on a sphere with two handles, Fig. 12 (b), i.e., a double torus, as shown in Fig. 12 (c). Modifications are made by four proper fillings and one perforation *P* of order 2, which are transferred to embedding as illustrated in Fig. 12 (d). The regional 4-sided cycles together with additional 4 shaded cycles minus the cycle corresponding to the perforation, form a cycle basis of *S*.

Seventy-nine cycles of length 4 and two cycles of length 8 are selected as a minimal cycle basis of *S*. However, for multi-member complex structures, this method is by no means easy to perform.

Example 3: Let *S* be the graph model of 3-story frame as shown in Fig. 13 (a). The first story can be embedded on a torus and the third story on a second torus. Four tubes can be added to accommodate the columns of the second story. Identifying these through one of the tubes results in a sphere with five handles as depicted in Fig. 13 (b). Once the embedding is achieved, modifications can easily be made by adding 5 × 2 = 10 fillings and one perforation. In the schematic representation fillings are shaded in red and the necessary perforation is denoted by *P*, Fig. 13 (b).

In a manifold embedding, the quality of the selected cycle basis depends upon the genus of the manifold on which *S* is embedded. Thus, it is ideal to have a minimal embedding. However, little is known about an efficient approach to carry out such an embedding.

4.4 Embedding S on a union of disks

S can be considered as the union of some planar connected non-separable subgraphs of *S*. De C. Henderson [14]. The process of such an embedding is as followings:

- Step 1: Identify a planar subgraph S_1 and embed it on a disk d_1 whose dissection K_1 is isomorphic to S_1.
- Step 2: The second subgraph S_2 is identified such that the corresponding K_2 has a 2-cell with a free and $|K_1| \cap |K_2|$ is a connected subspace of the frontier of d_1.
- Step K_i is joined to $K^{i-1} = \bigcup_{j=1}^{i-1} K_j$ with K_i having a free Step q: The process Step 2 is continued and at the i^{th} 1-face and $|K_i| \cap |K^{i-1}|$ being a connected subspace of the frontier of d_{i-1}. Obviously K^i is collapsible to K^{i-1}.

The process will be terminated when all 1-cells of *S* are embedded in $K^q = \bigcup_{j=1}^{q} K_j$ which is collapsible, since

$$K = K^q \to K^{q-1} \to \ldots \to K^i \to \ldots \to K^1 = K_1. \tag{13}$$

Example 1: Consider a space frame *S* as illustrated in Fig. 14. This model is embedded on three disks K_1–K_3 as shown in Fig. 14, resulting in a cycle basis consisting of 56 regional cycles of length.

It is ideal to embed *S* in a complex *K* with a minimum number of disks for all possible collapsible embeddings.

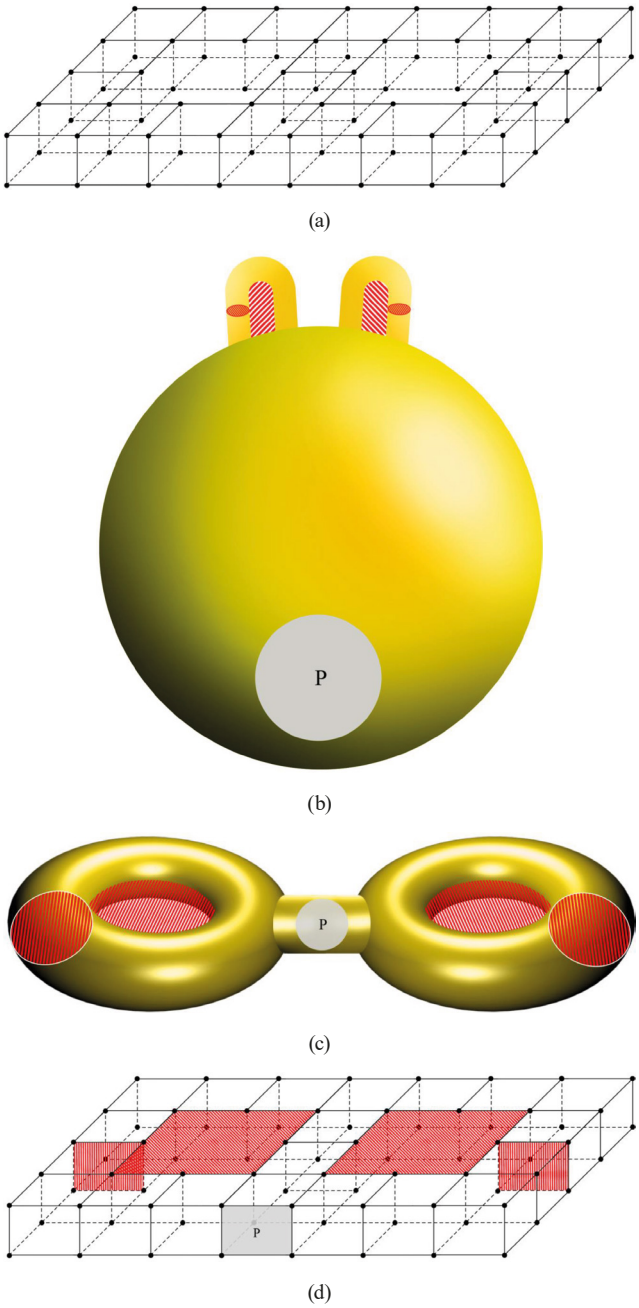


Fig. 12 An admissible embedding of *S*: (a) *S*, (b) a sphere with two handles and one perforation, (c) schematic illustration of the modified torus, (d) *S* and its embedding

This number is known as *thickness* of the graph *S*, denoted by $t(S)$. Hence an embedding should be performed with q as the thickness of *S*. Again, only partial results are available about the thickness of graphs. Any approach to embedding on a smaller number of disks would be advantageous for reducing the overlaps of the cycles.

Remark: The computer implementation of the methods of this section for selecting a cycle basis may become uneconomical from engineering point of view, however, the study of these methods provides a firm background for

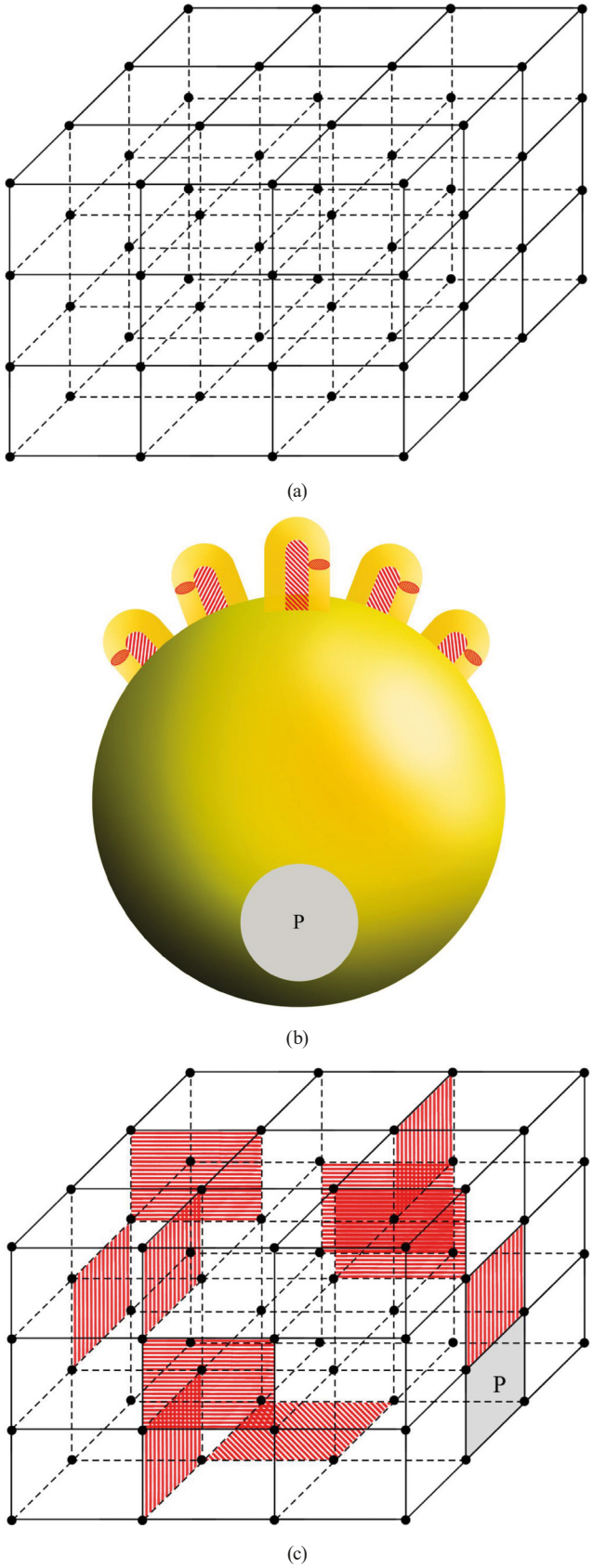


Fig. 13 A modified manifold embedding of a space graph: (a) graph model of a space frame, (b) a sphere with five handles, (c) schematic illustration of the embedding

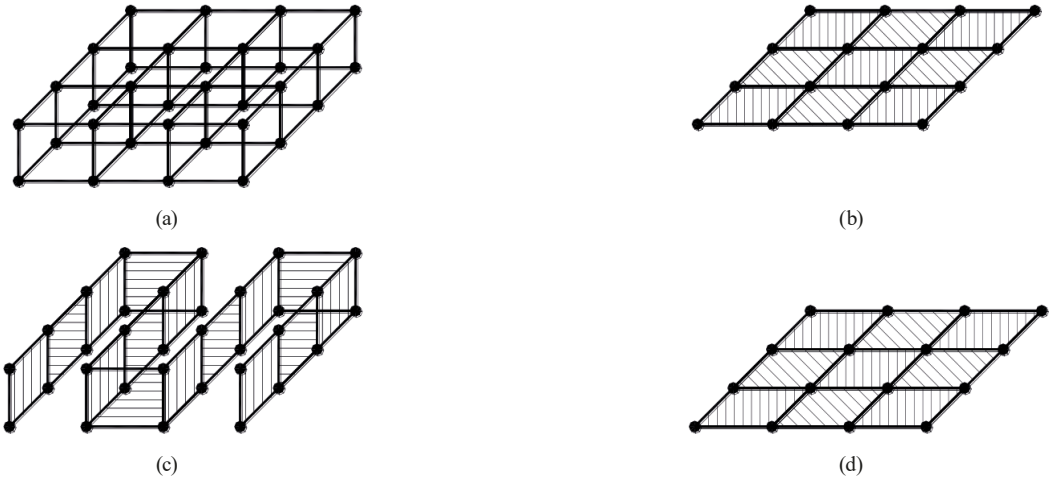


Fig. 14 A space graph and the identified disks K_1–K_3: (a) S, (b) K_1, (c) K_2, (d) K_3

designing many graph theoretical approaches. The visualization of these methods helps to develop algorithms having different characteristics. As an example, the expansion process may be viewed as an embedding of a graph on the elements of a disk space, with disks having certain properties. Similarly, one can design an algorithm for embedding a graph on the elements of a ball space, with balls having specified properties (the reverse process of a collapsible embedding).

Different graph theoretical methods for design of different efficient algorithms are efficiently implemented, Kaveh [13]. In this paper, algorithms are presented for selection of subminimal, and minimal, cycle bases of graphs. The interested reader many refer to Kaveh [13, 24] where suboptimal cycle bases are also selected. Method was applied to non-linear analysis of frame structures by [25], and generalization of the methods for the formation of generalized cycle bases can be found in Kaveh [26].

Force method is applied to structural metaheuristic optimization algorithms in [27, 28], finite element analysis in [29, 30], and it is utilized in the analysis of regular structures by [31, 32]. Many interesting applications like plastic shakedown analysis, reliability and topology optimization are recently published in [33–38].

5 Graph concepts for the force method

In this section, concepts and algorithms from graph theory are provided which are used for the force method of structural analysis. The force method can be categorized into five types, as mentioned in the introduction. Kaveh [13] demonstrated the efficiency of the graph-theoretical force method compared to the other type of force method. Therefore, the graph-based force method is further elaborated.

5.1 Minimal and optimal cycle bases

A matrix is called *sparse* if many of its entries are zero. The interest in sparsity arises because its exploitation can lead to considerable computational saving, and because many large matrices that occur in the analysis of practical structures, can be made sparse if they are not already so. A matrix can therefore be considered sparse, if there is an advantage in exploiting its zero entries.

The *sparsity coefficient* χ of a matrix is defined to be its number of non-zero entries. A cycle basis $C = \{C_1, C_2, C_3, \ldots, C_{b_1(S)}\}$ is called *minimal*, if it corresponds to a minimum value of

$$L(C) = \sum_{i=1}^{b_1(S)} L(C_i). \tag{14}$$

Obviously, $\chi(\boldsymbol{C}) = L(C)$ and a minimal cycle basis can be defined as a basis which corresponds to minimum $\chi(\boldsymbol{C})$. A cycle basis for which $L(C)$ is near minimum is called a subminimal cycle basis of S.

A cycle basis corresponding to maximal sparsity of the $\boldsymbol{C}\boldsymbol{C}^t$ is called an *optimal* cycle basis of S. If $\chi(\boldsymbol{C}\boldsymbol{C}^t)$ does not differ considerably from its minimum value, then the corresponding basis is termed *suboptimal*.

The matrix intersection coefficient $\sigma_i(\boldsymbol{C})$ of row i of cycle member incidence matrix $\boldsymbol{C}$ is the number of row j such that:

1. $j \in \{i + 1, i + 2, \ldots, b_1(S)\}$,
2. $C_i \cap C_j \neq \varnothing$, i.e., there is at least one k such that the column k of both cycles C_i and C_j (rows i and j) contain non-zero entries.

Now it can be shown that:

$$X(\boldsymbol{D}) = b_1(S) + 2\sum_{i=1}^{b_1(S)-1} \sigma_i(\boldsymbol{C}). \tag{15}$$

This relationship shows the correspondence of a cycle member incidence matrix $\boldsymbol{C}$ and that of its cycle basis adjacency matrix. In order to minimize $\chi(\boldsymbol{CC}^t)$, the value of $\sum_{i=1}^{b_1(S)-1} \sigma_i(\boldsymbol{C})$ should be minimized, since $b_1(S)$ is a constant for a given structure S, i.e., γ-cycles with a minimum number of overlaps should be selected.

In the force method, an optimal cycle basis is needed corresponding to the maximum sparsity of $\boldsymbol{CC}^t$ matrix. However, due to the complexity of this problem, most of the research has been concentrated on subminimal cycle basis selection, except few works [9, 13], which try to minimize the overlaps of the cycles rather than only minimizing their lengths.

5.2 Selection of minimal and subminimal cycle bases

Formation of a minimal cycle on a member: A minimal length cycle C_i on a member m_j, called its *generator*, can be formed by using the shortest route tree algorithm as follows:

- Start the formation of two SRTs rooted at the two end nodes n_s and n_t of m_j, and terminate the process as soon as the SRTs intersect each other (not through m_j itself) at say n_c. The shortest paths between n_s and n_c, and n_t and n_c, together with m_j, form a minimal cycle C_i on m_j. Using this algorithm, cycles of prescribed lengths can also be generated.
- As an example, C_i is a minimal cycle on m_j in Fig. 15. The SRTs are shown in bold lines. The generation of SRTs starting from n_s and n_t is terminated as soon as n_c has been found.
- A minimal cycle on a member m_j passing through a specified node n_k can similarly be generated. An SRT rooted at n_k is formed and as soon as it hits the end nodes of m_j, the shortest paths are found by backtracking between n_k and n_s, and n_k and n_t. These paths together with m_j form the required cycle. As an example, a minimal cycle on m_j containing n_k, is illustrated by dashed lines in Fig. 15.

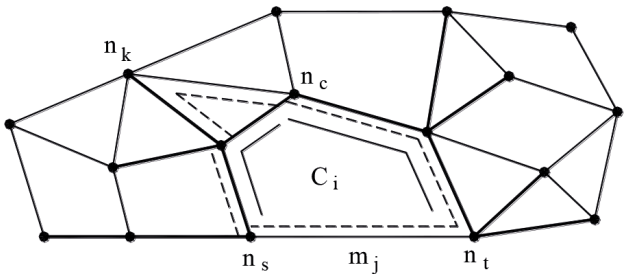


Fig. 15 A minimal cycle on a member

Different cycle sets for selecting a cycle basis: It is obvious that a general cycle can be decomposed into its simple cycles. Therefore, it is natural to limit the considered set to only simple cycles of S. Even such a cycle set, which forms a subspace of the cycle space of the graph, has many elements for large graphs and is therefore uneconomical for practical purposes.

In order to overcome the above difficulty, Kaveh [17] used an expansion process, selecting the smallest admissible (independent with additional restriction) cycles, one at a time, until $b_1(S)$ cycles forming a basis had been obtained. In this approach, a very limited number of cycles were checked for being an element of a basis. As an example, the expansion process for selecting a cycle basis of S is illustrated in Fig. 16.

Hubicka and Syslø [7] employed a similar approach, without the restriction of selecting one cycle at each step of expansion. In their method, when a cycle has been added to the previously selected cycles, increasing the first Betti number of the expanded part by "p", then p created cycles have been formed. As an example, in this method, Steps 4 and 5 will be combined into a single step, and addition of cycle 5 will require immediate formation of the cycle 4. The above method is modified, and an efficient algorithm is developed for the formation of cycle bases by Kaveh and Roosta [10].

Finally, Horton [11] proved that the elements of a minimal cycle basis lie in between a cycle set consisting of the minimal cycles on each member of S which passes through each node of S, i.e., each member is taken in turn and all cycles of minimal length on such a member passing through all the nodes of S are generated.

Independence control: Each cycle of a graph can be considered as a column vector of its cycle-member incidence matrix. An algebraic method such as the Gaussian elimination may then be used for checking the independence of a cycle with respect to the previously selected sub-basis. However, although this method is general and reduces the order dependency of the cycle selection algorithms, like many other algebraic approaches its application requires a considerable amount of storage space.

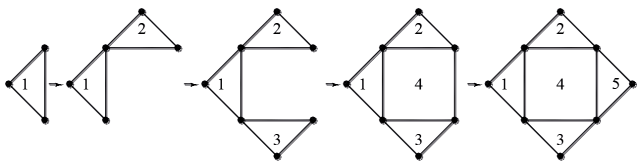


Fig. 16 A graph S and the selected cycles

The most natural graph theoretical approach is to employ a spanning tree of S, and form its fundamental cycles. This method is very simple; however, in general its use leads to long cycles. The method can be improved by allowing the inclusion of each used chord in the branch set of the selected tree. Further reduction in length may be achieved by generating an SRT from a center node of a graph, and the use of its chords in ascending order of distance from the center node, Kaveh [17].

A third method, which is also graph-theoretical, consists of using admissible cycles. Consider the following expansion process, with S being a 2-connected graph,

$$C_1 = C^1 \to C^2 \to C^3 \to \ldots \to C^{b_1(S)} = S, \tag{16}$$

where $C^k = \bigcup_{i=1}^{k} C_i$. A cycle C_{k+1} is called an *admissible* cycle, if for $C^{k+1} = C^k \cup C_{k+1}$:

$$b_1\left(C^{k+1}\right) = b_1\left(C^k \cup C_{k+1}\right) = b_1\left(C^k\right) + 1. \tag{17}$$

It can easily be proved that the above admissibility condition is satisfied if any of the following conditions hold:

1. $A_{k+1} = C^k \cap C_{k+1} = \varnothing$, where $\varnothing$ is an empty intersection;
2. $\bar{b}_1\left(A_{k+1}\right) = r - s$, where r and s are the numbers of components of C^{k+1} and C^k, respectively. In order to avoid counting the number of components one can connect C^k to the subsequently selected disjoint cycle C_{k+1} by a shortest path.
3. $\bar{b}_1\left(A_{k+1}\right) = 0$ when C^k and C_{k+1} are connected ($r = s$).

In the above relations, $\bar{b}_1\left(A_i\right) = \bar{M}_i - \bar{N}_i + 1$, where $\bar{M}_i$ and $\bar{N}_i$ are the numbers of members and nodes of A_i, respectively.

As an example, the sequence of selecting cycles in Fig. 17 will be as specified by their numbers.

A different approach suggested by Hubicka and Syslø [7], in which,

$$b_1\left(C^{k+1}\right) = b_1\left(C^k\right) + p, \tag{18}$$

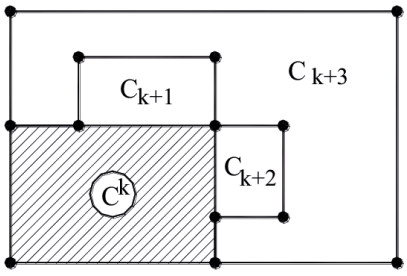


Fig. 17 A cycle and its bounded cycles

is considered to be permissible. However, a completion is performed for $p > 1$. As an example, when C_3 is added to C^k, its first Betti number is increased by 3 and therefore, cycles C_{k+1} and C_{k+2} must also be selected at that stage, before further expansion.

Having discussed the mathematical concepts involved in a cycle basis selection, three different algorithms are described in Algorithms 1–3.

The method of Algorithm 1 selects subminimal cycle bases, using the chords of an SRT. The nodes and members of the tree and consequently the cycles are partially ordered according to their distance from O. This is

Algorithm 1 An SRT based subminimal cycle basis (Kaveh [17])

Step 1: Select a pseudo-center node of maximal degree O. Such a node can be selected manually or automatically using the graph or algebraic graph theoretical methods.

Step 2: Generate an SRT rooted at O, form the set of its chords and order them according to their distance from O.

Step 3: Form one minimal cycle on each chord in turn, starting with the chord nearest to the root node. A corresponding simple path is chosen which contains members of the tree and the previously used chords, hence providing the admissibility of the selected cycle.

Algorithm 2 An expansion based subminimal cycle basis (Kaveh [17])

Step 1: Select a center or pseduo-center node of maximal degree O.

Step 2: Use any member incident with O as the generator of the first minimal cycle. Take any member not used in C_1 and incident with O, and generate on it the second minimal cycle. Continue this process until all the members incident with O are used as the members of the selected cycles. The cycles selected so far are admissible, since the intersection of each cycle with the previously selected cycles is a simple path (or a single node) resulting in an increase of the first Betti number by unity for each cycle.

Step 3: Choose a starting node O′, adjacent to O, which has the highest degree. Repeat a step similar to Step 2, testing each selected cycle for admissibility. If the cycle formed on a generator m_k fails the test, then examine the other minimal cycles on m_k if any such cycle exists. If no admissible minimal cycle can be found on m_k, then,

Form admissible minimal cycles on the other members incident with O′. If m_k does not belong to one of these subsequent cycles, then:

Search for an admissible minimal cycle on m_k, since the formation of cycles on other previous members may have now altered the admissibility of this cycle. If no such cycle can be found, leave m_k unused. In this step more than one member may be left unused.

Step 4: Repeat Step 3 using as starting nodes a node adjacent to O and/or O′, having the highest degree. Continue the formation of cycles until all the nodes of S have been tested for cycle selection. If all the members have not been used, select the shortest admissible cycle available for an unused member as generator. Then test the minimal cycles on the other unused members, in case the formation of the longer cycle has altered the admissibility. Each time a minimal cycle is found to be admissible, add to C^i and test all the minimal cycles on the other unused members again. Repeat this process, forming other shortest admissible cycles on unused members as generators, until S is re-formed and a subminimal cycle basis has been obtained.

Algorithm 3 An expansion using cycle length (Kaveh [4])

Step 1: Generate as many admissible cycles of length 3 as possible. Denote the union of the selected cycles by C^n.

Step 2: Select an admissible cycle of length 4 on an unused member. Once such a cycle C_{n+1} is found, check the other unused members for possible admissible cycles of length 3. Again select an admissible cycle of length 4 followed by the formation of possible 3-sided cycles. This process is repeated until no admissible cycles of length 3 and 4 can be formed. Denote the generated cycles by C^m.

Step 3: Select an admissible cycle of length 5 on an unused member. Then check the unused members for the formation of 3-sided admissible cycles. Repeat Step 2 until no cycle of length 3 or 4 can be generated. Repeat Step 3 until no cycle of length 3, 4 or 5 can be found.

Step 4: Repeat similar steps to Step 3, considering higher-length cycles, until $b_1(S)$ admissible cycles forming a subminimal cycle basis are generated.

the combinatorial version of the Turn Back method to be discussed in the selection of null basis by algebraic force methods.

Algorithms 1 and 2 are order-dependent, and various starting nodes may alter the result slightly. Algorithm 3 is more flexible and less order-dependent, and in general leads to the formation of shorter cycle bases.

Remark 1: The cycle basis C formed by Algorithms 1–3 can further be improved by exchanging the elements of the selected basis. In each step of this process, a shortest cycle C_i' independent of the cycles of $C \backslash C_i$ is replaced by C_i if $L(C_i') < L(C_i)$. This process is repeated for $i = 1, 2, \ldots, b_1(S)$.

This additional operation increases the computational time and storage, and its use is recommended only when the formation of a minimal cycle basis is required.

Remark 2: In order to avoid counting the number of components if C_{k+1} is disjoint from C^k, one can connect C^k to the subsequently selected cycle C^{k+1} by a shortest path, to keep C^{k+1} as connected subgraph.

A simplified version of Algorithm 4 can be designed in Algorithm 5.

Algorithm 4 Minimal cycle basis (Horton [11])

Step 1: Find a minimum path $P(n_i, n_j)$ between each pair of nodes n_i and n_j.

Step 2: For each node nk and member $m_1 = (n_i, n_j)$, generate the cycle having m_1 and n_k as $P(n_k, n_i) + P(n_k, n_j) + (n_i, n_j)$ and calculate its length. Degenerate cases in which $P(n_k, n_i)$ and $P(n_k, n_j)$ have nodes other than n_k in common, can be omitted.

Step 3: Order the cycles by their weight (or length).

Step 4: Use the Greedy Algorithm, to find a minimal cycle basis from this set of cycles.

Algorithm 5 Simplified Horton

Step 1: Form a spanning tree of S rooted from an arbitrary node, and select its chords.

Step 2: Take the first chord and form $N(S) - 2$ minimal cycles, each being formed on the specified chord containing a node of S (except the two end nodes of this chord).

Step 3: Repeat Step 2 for the other chords, in turn, until $[M(S) - N(S) + 1] \times [N(S) - 2]$ cycles are generated. Repeated and degenerate cycles should be discarded.

Step 4: Order the cycles in ascending magnitude of their lengths.

Step 5: Using the above set of cycles, employ the Greedy Algorithm to form a minimal cycle basis of S.

The main contribution of Horton's Algorithm is the limit imposed on the elements of the cycle-set used in the Greedy Algorithm. The use of matroids and the Greedy Algorithm, has already been suggested by Kaveh [4], and they have been employed by Kolasińska [8].

6 Examples for studying special features

The following examples are designed to illustrate special features which are used in the process of the Algorithms 1–3.

Example 1: Consider a planar graph S, as shown in Fig. 18, having $b_1(S) = 18 - 11 + 1 = 8$. Using Algorithm 3, the selected basis consists of four cycles of length 3, three cycles of length 4 and one cycle of length 5, as follows:

$$C_1 = (1,2,3),\ C_2 = (1,8,9),\ C_3 = (2,6,3),\ C_4 = (2,5,6),$$
$$C_5 = (1,4,5,2),\ C_6 = (1,7,5,2),\ C_7 = (8,6,2,1),$$
$$C_8 = (10,8,6,3,11).$$

The total length of the selected basis is $L(C) = 29$, which is a counter example for minimality of a mesh basis, since, for any such basis of S, $L(C) > 29$.

Example 2: In this example, S is the model of a space frame, considered as $S = \bigcup_{i=1}^{27} S_i$, where a typical S_i is

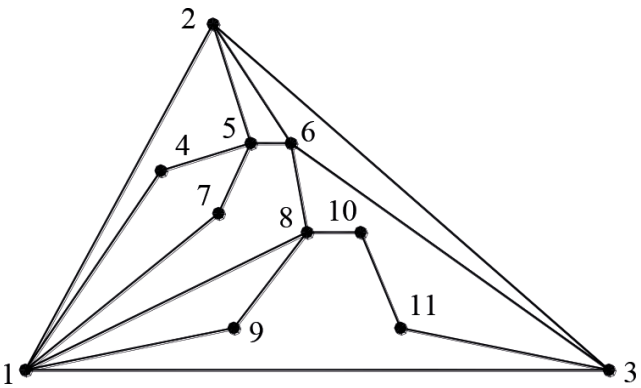


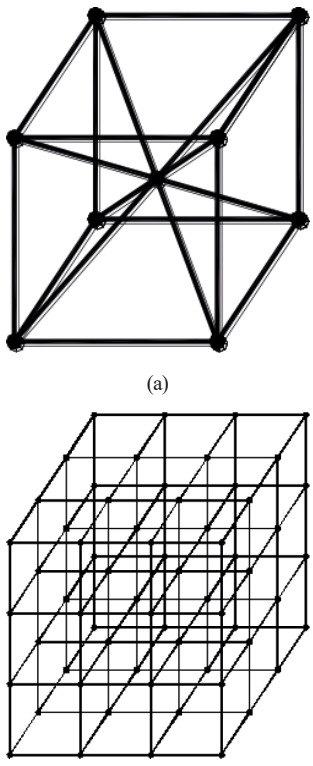
Fig. 18 A planar graph S

depicted in Fig. 19 (a). For S_i there are 12 members joining a central node to the 8 corner nodes. The model S is shown in Fig. 19 (b), in which some of the members are omitted for clarity in the diagram. For this graph, $b_1(S) = 270$.

The selected cycle basis using any of the Algorithms 1–3 consists of 270 cycles of length 3, forming a minimal cycle basis of S. For Algorithm 3, the use of different starting nodes leads to a minimal cycle basis, indicating the capability of this method.

Example 3: S is a planar graph with $b_1(S) = 9$, as shown in Fig. 20. The application of Algorithm 3 results in the formation of a cycle of length 3 followed by the selection of five cycles of length 4. Then the member {1,6} is used as the generator of a six-sided cycle C_7 = (1,2,3,4,5,6,1). Member {2,10} is employed to form a seven-sided cycle C_8 = (2,11,12,13,14,15,10,2), followed by the selection of a five-sided cycle C_9 = (10,5,4,3,2,10). The selected cycle basis has a total length of $L(C)$ = 41, and is not a minimal cycle basis. A shorter cycle basis can be found by Algorithm 4 consisting of one three-sided and five four-sided cycles, together with the following cycles,

$C_7 = (1,2,10,5,6,1)$, $C_8 = (2,3,4,5,10,2)$ and
$C_9 = (2,11,12,13,14,15,10,2)$,



(a)

(b)

Fig. 19 A space frame S: (a) a typical S_i(i = 1, …, 27), (b) S with some omitted members

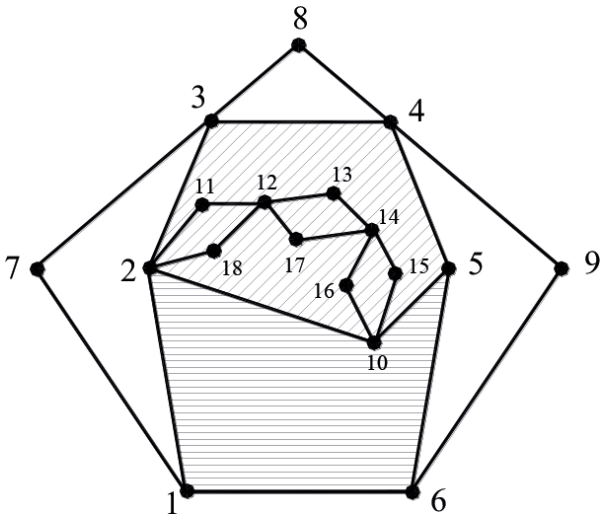


Fig. 20 A planar graph S

forming a basis with the total length of 40. However, the computation time and storage for Algorithm 3 is far less than that of Algorithm 4, as compared in [18].

7 Formation of $\boldsymbol{B}_1$ matrices

The structure of the matrix $[\boldsymbol{B}_1]$ is important because it should be inverted in the solution. This matrix has $6M$ rows and $6b_1(S)$ columns, with $b_1(S)$ being the first Betti number defined as:

$$b_1(S) = M(S) - N(S) + b_0(S), \tag{19}$$

where, $b_0(S)$ is the number of components of the structural model.

In the graph-theoretical force method, first the cycle basis of the structure is formed using one of the available methods, then $[\boldsymbol{B}_1]$ is calculated utilizing the elements of the selected cycle basis. The term "generator" refers to one of the members of a considered cycle. The generator is cut in the neighborhood of its beginning node, and six bi-actions are applied. In the sub-matrix of $[\boldsymbol{B}_1]_{ij}$, the columns contain the internal forces at the lower-numbered end of the i^{th} member under the application of six bi-actions at the cut of the j^{th} generator, as given in the following:

$$\boldsymbol{B}_1 = \begin{matrix} \\ i \end{matrix} \overset{j}{\begin{bmatrix} & [\boldsymbol{B}_1]_{ij} & \\ & & \end{bmatrix}}. \tag{20}$$

in which

$$[\boldsymbol{B}_1]_{ij} = \beta_{ij} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\Delta z & \Delta y & 1 & 0 & 0 \\ \Delta z & 0 & -\Delta x & 0 & 1 & 0 \\ -\Delta y & \Delta x & 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$\Delta x_{B1} = x_j - x_k$$
$$\Delta y_{B1} = y_j - y_k$$
$$\Delta z_{B1} = z_j - z_k$$

$$\beta_{ij} = \begin{cases} +1 & \text{if member } i \text{ has the same orientation of the cycle generated on } j, \\ -1 & \text{if member } i \text{ has the reverse orientation of the cycle generated on } j, \\ 0 & \text{if member is not in the cycle whose generator is } j. \end{cases}$$

Here, x_j, y_j, and z_j are the coordinates of the beginning node of the generator j, and x_k, y_k, and z_k are the coordinates of the lower numbered of the i^{th} member. More detail is available in [13, 15].

8 Conclusions

This paper shows how some concept from topology can be utilized for embedding the graphs on higher dimensional topological spaces, leading to efficient graph theoretical methods for the selection of different cycle bases, which are more suitable to computer implementation. Self-equilibrating stress systems forming efficient static bases can then be formed on the selected cycles leading to highly sparse flexibility matrices. The presented methods show their power in both multi-member (real-size) structures and also in optimal design of structures where efficient force method can be performed in place of displacement approach. The cycles can be ordered to form banded flexibility matrices. Conditioning of the flexibility matrices can also be improved considering the material and geometric properties of the elements and selecting cycles with strong overlap. Thus, the conditions for performing the optimal analysis of frame structures can be fulfilled.

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