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A Localized Version of the Method of Fundamental Solutions in a Multi-level Context

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Abstract

The Method of Fundamental Solutions is applied to the Laplace equation. Instead of using the traditional approach with external source points and boundary collocation points, the original domain decomposed into a lot of smaller, overlapping subdomains, and the Method of Fundamental Solutions is used to the individual local subdomains. After eliminating the local source points, local schemes are obtained. Instead of constructing a global scheme, the local subproblems are solved sequentially, in an iterative way. This mimics a multiplicative Schwarz method with overlapping subdomains, which assures the convergence of the method. Combining the iteration with a simple Seidel-type method, the resulting iteration is used as a smoothing procedure of a multi-level method. The points belonging to the coarse and fine levels are defined by a quadtree-generated cell system controlled by the boundary of the original domain. The multi-level character of the obtained method makes it possible to reduce the necessary number of iterations, that is, the overall computational cost can be significantly reduced. Moreover, the solution of large and ill-conditioned systems is completely avoided. The method is illustrated through several numerical test examples.

Keywords

method of fundamental solutions, local schemes, multi-level methods, quadtrees

1 Introduction

Elliptic partial differential equations are still important mathematical tools to describe a lot of steady physical phenomena and engineering problems. There are traditional computational techniques to approximately solve these partial differential equations such as the well-known finite difference methods or the finite element methods. However, these methods require complicated computational grid or mesh structures, the construction of which is often difficult and time consuming. In contrast to the above mesh-based methods, *meshless methods* have quickly been quite popular since they require neither domain nor boundary mesh or grid structure. Instead, a scattered point set is required on the boundary and/or in the domain of the partial differential equation to be solved.

The Method of Fundamental Solution (MFS, see e.g., [1, 2]) is a special meshless method, which can be applied to a lot of engineering problems, e.g., potential problems or some simple transport problems, Stokes flow problems, Helmholtz problems etc. Suppose that the original elliptic partial differential equation has the form:

$$Lu(x) = 0 \quad (x \in \Omega), \tag{1}$$

which is defined in a multidimensional domain (i.e., an open and connected set) Ω and is supplied with a usual boundary condition (i.e., Dirichlet or Neumann or mixed boundary condition). Then the MFS produces the solution of the original problem in the following form:

$$u(x) = \sum_{j=1}^{M} \alpha_j \cdot \Phi(x - s_j). \tag{2}$$

Here Φ denotes the fundamental solution of the original differential operator L, therefore the function u defined by Eq. (2) exactly satisfies the partial differential Eq. (1). The numbers $\alpha_1, \ldots, \alpha_M$ are a priori unknown coefficients, $s_1, \ldots, s_M$ are predefined points in the exterior of the domain Ω (*source points*). The coefficients $\alpha_1, \ldots, \alpha_M$ are computed by enforcing the boundary conditions at some predefined $x_1, \ldots, x_N$ *boundary collocation points*. This results in a linear system of equations with M unknowns and N equations. If the simplest Dirichlet boundary condition is prescribed, this linear system has the form:

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$$\sum_{j=1}^{M} \alpha_j \cdot \Phi\left(x_k - s_j\right) = u\left(x_k\right), \tag{3}$$

where k =1, 2, … , N. The numbers M and N need not be equal. If they differ, a least squares technique or the Singular Value Decomposition can be used.

Though the MFS is a simple, truly meshless, and easy-to-program method, it has several disadvantages. First, the fundamental solution must be explicitly known. Some additional problems are as follows: the location of the source points is not trivial, no optimal arrangement is known; even if the numbers of source and collocation points are equal, the system Eq. (2) may be severely ill-conditioned, especially when the sources are located too far from the boundary. On the other hand, if they are too close to the boundary, numerical singularities may occur, which increases the error of the approximation etc.

To overcome these difficulties, a number of special techniques have been developed. A wide class of these methods is based on allowing the source points and boundary collocation points to coincide. This approach has to handle the appearing singular terms in a proper way (regularization, desingularization, see e.g., [3, 4]); moreover, the resulting system remains dense and ill-conditioned. It is also possible to use *several* groups of sources at the same time, the spatial densities of which decrease far from the boundary. This idea can be embedded in a multi-level context, which significantly reduces the necessary computational cost. For details, see [5, 6].

In this paper, a combination of the Method of Fundamental Solutions and a localization technique is presented. The technique provides a global system with sparse matrix. This system of equations is solved by a special iterative way, which can be embedded in a multi-level context: the coarse and fine levels are defined by a quadtree-based subdivision technique.

For the sake of simplicity, the method is introduced through the example of the 2D Laplace equation. The generalization for higher dimensional problems is straightforward. The same technique can be applied also to more general elliptic partial differential equations provided that the fundamental solution is explicitly known.

2 Localization techniques

A familiar strategy to handle elliptic problems is the use of *local schemes*. The well-known finite difference method was the prototype of such techniques, which is traditionally based on the Taylor series expansion using a (Cartesian or curvilinear) grid structure. However, local schemes can be constructed also in *meshless* way based on a scattered data interpolation technique.

Consider a central point $x_C^{(0)}$ in Ω, the domain of the original partial differential equation and some neighboring points $x_C^{(1)}$, …, $x_C^{(N_C)}$. Denote by $u_C^{(0)}$ (and $u_C^{(1)}$, …, $u_C^{(N_C)}$, respectively) the values of the approximate solution u at these points. Define an interpolation function $\tilde{u}$ as follows:

$$\tilde{u}\left(x\right) = \sum_{j=0}^{N_C} \alpha_j \cdot \Psi\left(x - x_C^{(j)}\right), \tag{4}$$

where Ψ is a predefined radial basis function e.g., the thin plate spline: $\Psi(x) = \|x\|^2 \cdot \log\|x\|$ (here $\|\cdot\|$ denotes the Euclidean norm in the two-dimensional space). The coefficients α_0, …, α_{N_C} are determined by the interpolation conditions (Eq. (5)).

$$\sum_{j=0}^{N_C} \alpha_j \cdot \Psi\left(x_C^{(k)} - x_C^{(j)}\right) = u_C^{(k)} \tag{5}$$

(k = 0, … ,N_C), which results in a (local) system of equations. After solving this system, the interpolated values of u at a part of a *structured* grid around $x_C^{(0)}$ can be computed, and familiar finite difference schemes can be constructed without any difficulty. For instance, the Laplacian of u can be approximated by the usual 5-point scheme as illustrated in Fig. 1:

$$\left(\Delta u\right)_C \sim \frac{1}{h^2} \cdot \left(\tilde{u}_N + \tilde{u}_W + \tilde{u}_S + \tilde{u}_E - 4u_C^{(0)}\right),$$

where the fictitious points $\tilde{x}_N$, $\tilde{x}_W$, $\tilde{x}_S$, $\tilde{x}_E$ are defined to be the neighbors of the central point $x_C^{(0)}$ taken in the main coordinate directions at a distance h from the central point. The values $\tilde{u}_N$, $\tilde{u}_W$, $\tilde{u}_S$, $\tilde{u}_E$ are computed by interpolation: $\tilde{u}_N := \tilde{u}(\tilde{x}_N)$ … etc. See Fig. 1 for illustration.

The technique can easily be combined with a Seidel-type iteration in a natural way. For the 2D Laplace equation, this result in the following local scheme (for details, see [7]):

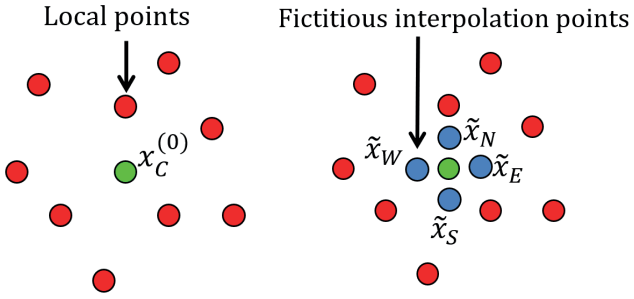


Fig. 1 Local points (red) and local interpolation to the fictitious neighboring points $\tilde{x}_N$, $\tilde{x}_W$, $\tilde{x}_S$, $\tilde{x}_E$

$$u_C^{(0)} := \frac{1}{4}\cdot\left(\tilde{u}_N + \tilde{u}_W + \tilde{u}_S + \tilde{u}_E\right). \tag{6}$$

More recently, a localization strategy based on the Method of Fundamental Solutions has been developed [8]. Instead applying a local interpolation, for each local subdomain, a (small) set of local source points $s_C^{(1)}$, …, $s_C^{(M_C)}$ is defined (e.g., along the perimeter of a circle centered at $x_C^{(0)}$, see Fig. 2), and the approximate local solution is sought in the MFS-form:

$$u(x) = \sum_{j=1}^{M_C} \alpha_j \cdot \Phi\left(x - s_C^{(j)}\right), \tag{7}$$

where Φ denotes again the fundamental solution of the original problem.

The coefficients α_1, …, α_{M_C} are computed by enforcing the equalities Eq. (8).

$$\sum_{j=1}^{M_C} \alpha_j \cdot \Phi\left(x_C^{(k)} - s_C^{(j)}\right) = u\left(x_C^{(k)}\right) \tag{8}$$

($k = 1, \ldots, N_C$), and the central value $u_C^{(0)}$ is updated by:

$$u_C^{(0)} := u\left(x_C^{(0)}\right). \tag{9}$$

The system Eq. (8) is solved in the sense of least squares. This procedure results in a global system for the central values $u_C^{(0)}$, when the central point $x_C^{(0)}$ runs over the internal points. It should be pointed out that the matrix of the global system is sparse, which makes it possible to apply computationally efficient solution algorithms. Note however, that the validity of the equalities of the system Eq. (8) are not guaranteed theoretically: the collocation points $x_C^{(1)}$, …, $x_C^{(N_C)}$ cannot be considered boundary collocation points of the subdomain, therefore the prescription of the values of the solution here does not result in a well-posed problem. In spite of this fact, excellent exactness was reached. The idea has been applied to various problems, such as transient convection-diffusion-reaction equations. For details, see [8, 9].

In this paper, another technique is presented. Instead of enforcing the equalities Eq. (8) at all neighboring points $x_C^{(k)}$ ($k = 1, 2, \ldots, N_C$) they are required along the boundary of some small subdomains only. This results in a well-posed problem and makes the technique similar to the classical alternating method of Schwarz [10]. Later, the approach will be embedded in a multi-level context in a natural way.

3 Localization based on overlapping subdomains

First, let us briefly recall the main ideas of the traditional Schwarz alternating method (also referred to as multiplicative Schwarz method). We restrict ourselves to the pure Dirichlet problem of the 2D Laplace equation.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, sufficiently smooth domain. Assume that Ω has a decomposition $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are non-empty (overlapping) subdomains with boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively (see Fig. 3). Here $\Gamma_1 := \overline{\partial\Omega_1 \cap \Omega_2}$ and $\Gamma_2 := \overline{\partial\Omega_1 \cap \Omega_2}$ (the overbar denotes the closure of the set). The overlap is called *weak*, if the distance of Γ_1 and Γ_2 is equal to 0. If this distance is strictly positive, then the overlap is called *strong*. Consider the Dirichlet problem:

$$\Delta U = 0 \text{ in } \Omega,\ U|_{\partial\Omega} = u_0\,. \tag{10}$$

The Schwarz alternating method approximates the solution of Eq. (10) by the following sequence of subproblems:

Subproblem 1:

$$\Delta U_{n+1/2} = 0 \text{ in } \Omega_1 \tag{11}$$

$$U_{n+1/2}|_{\partial\Omega_1\setminus\Gamma_1} = u_0|_{\partial\Omega_1\setminus\Gamma_1},\ U_{n+1/2}|_{\Gamma_1} = U_n|_{\Gamma_1} \tag{12}$$

Subproblem 2:

$$\Delta U_{n+1} = 0 \text{ in } \Omega_2 \tag{13}$$

$$U_{n+1}|_{\partial\Omega_2\setminus\Gamma_2} = u_0|_{\partial\Omega_2\setminus\Gamma_2},\ U_{n+1}|_{\Gamma_2} = U_{n+1/2}|_{\Gamma_2} \tag{14}$$

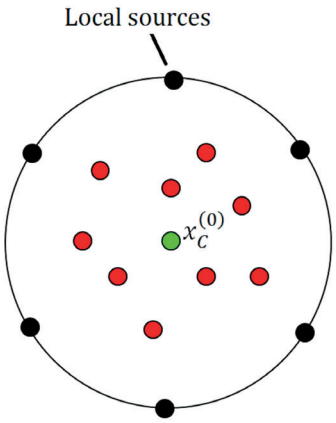
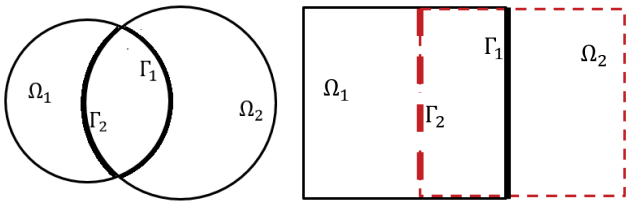


Fig. 2 Local sources (black points) along a circle



(a) (b)

Fig. 3 Weakly and strongly overlapping subdomains (a) weak overlap (Γ_1 and Γ_2 have common points), (b) strong overlap (the distance of Γ_1 and Γ_2 is positive)

Here $n = 1, 2, \ldots$ The iteration starts with an arbitrary initial approximation U_0 which satisfies the Laplace equation in Ω_2 and the boundary condition u_0 along $\partial\Omega_2\backslash\Gamma_2$.

Classical results guarantee that the sequence of the approximate solutions defined by the Schwarz alternating method converges to the exact solution of Eq. (10) (see e.g., [10, 11]) In the case of weak overlap, uniform convergence is achieved. Under the assumption of strong overlap, the Schwarz sequence converges with respect to the usual Sobolev norms. This can be shown by a standard technique based on Fourier series. For simplicity, assume that:

$$\Omega_1 = (-\pi + h, h) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ and}$$

$$\Omega_2 = (-h, \pi - h) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

with an overlap $2h$, where $0 < h < \pi/2$. Then:

$$\Gamma_1 = \{h\} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{and } \Gamma_2 = \{-h\} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Denote by $e_{n+1/2} := U_{n+1/2} - U$ and $e_{n+1} := U_{n+1} - U$, the errors of the approximations. Clearly:

$$\Delta e_{n+1/2} = 0 \text{ in } \Omega_1, \text{and } e_{n+1/2}|_{\partial\Omega_1\backslash\Gamma_1} = 0, \tag{15}$$

and similarly:

$$\Delta e_{n+1} = 0 \text{ in } \Omega_2, \text{and } e_{n+1}|_{\partial\Omega_2\backslash\Gamma_2} = 0. \tag{16}$$

Let us express $e_n|_{\Gamma_1}$ in terms of trigonometric Fourier series. Since e_n vanishes at the endpoints of Γ_1, it can be expressed as a sum of (shifted) pure sinusoidal terms:

$$e_n(h, y) = \sum_{k=1}^{\infty} \gamma_k \cdot \sin k\left(y + \frac{\pi}{2}\right). \tag{17}$$

The Fourier coefficients γ_k of course depend on the iteration index n; however, the notation of this dependence is omitted for the sake of simplicity. In the subdomain Ω_1:

$$e_{n+1/2}(x, y) = \sum_{k=1}^{\infty} \gamma_k \cdot \sin k\left(y + \frac{\pi}{2}\right) \cdot \frac{\sinh\left(k \cdot (x + \pi - h)\right)}{\sinh(k \cdot \pi)}, \tag{18}$$

therefore:

$$e_{n+1/2}|_{\Gamma_2} = e_{n+1/2}(-h, y) = \sum_{k=1}^{\infty} \gamma_k \cdot \sin k\left(y + \frac{\pi}{2}\right) \cdot \frac{\sinh\left(k \cdot (\pi - 2h)\right)}{\sinh(k \cdot \pi)}. \tag{19}$$

Thus, in the subdomain Ω_2, e_{n+1} can be expressed as:

$$e_{n+1}(x, y) = \sum_{k=1}^{\infty} \gamma_k \cdot \sin k\left(y + \frac{\pi}{2}\right) \cdot \frac{\sinh\left(k \cdot (\pi - 2h)\right)}{\sinh(k \cdot \pi)} \cdot \frac{\sinh\left(k \cdot (x - \pi + h)\right)}{\sinh(-k \cdot \pi)}, \tag{20}$$

therefore:

$$e_{n+1}|_{\Gamma_1} = e_{n+1}(h, y) = = \sum_{k=1}^{\infty} \gamma_k \cdot \sin k\left(y + \frac{\pi}{2}\right) \cdot \left(\frac{\sinh\left(k \cdot (\pi - 2h)\right)}{\sinh(k \cdot \pi)}\right)^2. \tag{21}$$

In short:

$$e_{n+1}(h, y) = \sum_{k=1}^{\infty} c_k^2 \cdot \gamma_k \cdot \sin k\left(y + \frac{\pi}{2}\right), \tag{22}$$

where $c_k \cdot \frac{\sinh\left(k \cdot (\pi - 2h)\right)}{\sinh(k \cdot \pi)}$. Standard calculations show that

$$0 \le c_k \le e^{-2kh}, \tag{23}$$

which means that all Fourier components of the error decrease at least by a factor e^{-4h} in each iteration step, and the higher frequency components decrease faster and faster. Consequently, the usual Sobolev norms of the error decrease at least by the above factor.

The method can be generalized for a finite covering of the original domain in a natural way. It should be pointed out, however, that the more subdomains are used, the slower overall convergence rate is achieved.

3.1 Localization and MFS-technique

The proposed method is based on the previously outlined Schwarz alternating method, which reduces the original problem to a sequence of problems defined on a subdomain of the original domain Ω (referred to as *local problems* henceforth). The local problems are solved by a standard MFS-based approach. Suppose that some local points are scattered in the original domain Ω. For an arbitrary central point $x_C^{(0)}$ and a predefined 'radius of influence' R, collect the neighboring local points, the distances of which from $x_C^{(0)}$ are sufficiently close to R, e.g., between $3R/4$ and R; denote them by $x_C^{(1)}, \ldots, x_C^{(N_C)}$. Define some local source points $s_C^{(1)}, \ldots, s_C^{(M_C)}$ along a circle centered at $x_C^{(0)}$ with radius greater than R (see Fig. 4). The local sources must be located in the exterior of the local subdomains: on the other hand, if they are too far from the boundary of the local subdomain, the local system of equations becomes highly ill-conditioned. As an acceptable compromise, the

radius of the circle containing the local sources was set to $2R$. The local solutions are defined by MFS-form augmented by harmonic polynomials:

$$u(x)=\sum_{j=1}^{M_C}\alpha_j\cdot\Phi\left(x-s_C^{(j)}\right)+\sum_{j=1}^{M}\beta_j\cdot p_j(x)\,, \tag{24}$$

where Φ denotes again the fundamental solution of the original problem, and $p_1, p_2,\ldots,p_M$ are the first M harmonic polynomials: $p_1(x,y)=1, p_2(x,y)=x, p_3(x,y)=y, p_4(x,y)=xy$, and so forth.

The coefficients $\alpha_1,\ldots,\alpha_{M_C}$ and $\beta_1,\ldots,\beta_M$ are computed by enforcing the equalities:

$$\sum_{j=1}^{M_C}\alpha_j\cdot\Phi\left(x_C^{(k)}-s_C^{(j)}\right)+\sum_{j=1}^{M}\beta_j\cdot p_j\left(x_C^{(k)}\right)=u(x_C^{(k)}), \tag{25}$$

$(k=1,2,\ldots,N_C)$, and possibly:

$$\sum_{j=1}^{M_C}\alpha_j\cdot p_k\left(s_C^{(j)}\right)=0,\quad (k=1,2,\ldots,M)\,. \tag{26}$$

Note that only the 'local boundary conditions' are enforced, i.e., at the local points, the distances of which from $x_C^{(0)}$ are between $3R/4$ and R (regarded as 'local boundary collocation points').

After solving a local subproblem (in the sense of least squares), update the value of the approximate solution at the central point $x_C^{(0)}$ by:

$$u_C^{(0)}:=\sum_{j=1}^{M_C}\alpha_j\cdot\Phi\left(x_C^{(0)}-s_C^{(j)}\right)+\sum_{j=1}^{M}\beta_j\cdot p_j\left(x_C^{(0)}\right). \tag{27}$$

This procedure mimics the Schwarz procedure at the subdomain (which is actually a circle centered at $x_C^{(0)}$ with radius R). It is also possible to update the values at all neighboring points located from the central point at a distance less than $3R/4$. For the sake of simplicity, however, we have updated the value of the central point only.

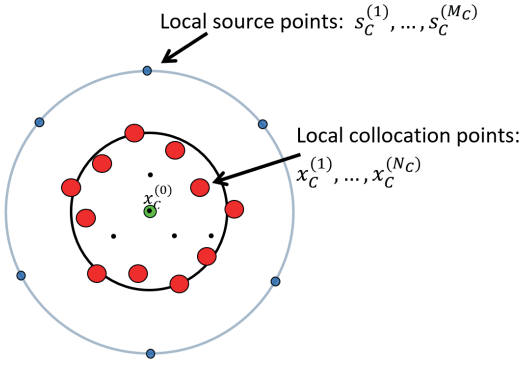


Fig. 4 Local source points (blue) and local collocation points (red). The internal black points are non-collocation local points

The number of local sources can be kept bounded. Therefore, the sizes of the local subproblems are small, so that the necessary number of arithmetic operations is moderate for each subproblem.

Due to the previous considerations, the resulting iterative method is convergent. However, the rate of convergence might be unsatisfactory. As pointed out earlier, the more subdomains (i.e., the smaller subdomains) are used, the lower the rate of convergence is achieved. This phenomenon is illustrated through the following example.

Example 1: Let Ω be the unit circle and consider the model problem

$$\Delta u=0 \text{ in } \Omega\,, \tag{28}$$

supplied with Dirichlet boundary condition consistent with the test solution:

$$u(x,y)=x^2-y^2\,. \tag{29}$$

In the interior of the domain, N local points were defined in a pseudo-random way. $M_C := 8$ local sources were defined in each local subdomain. The number of the applied harmonic polynomials was set to $M := 3$. Fig. 5 shows the relative discrete L_2-errors of the approximate solution with respect to the number of iterations. The total numbers of local points were: $N = 300$ (blue line), $N = 1200$ (green line) and $N = 4800$ (red line). (Since the model problem is a 2D one, the density of local points increases by a factor of 2 between the consecutive examples.) The corresponding 'radii of influence' were set to $R := 0.4$, $R := 0.2$, and $R := 0.1$, respectively. Thus, the numbers of neighbors of central points were in the same

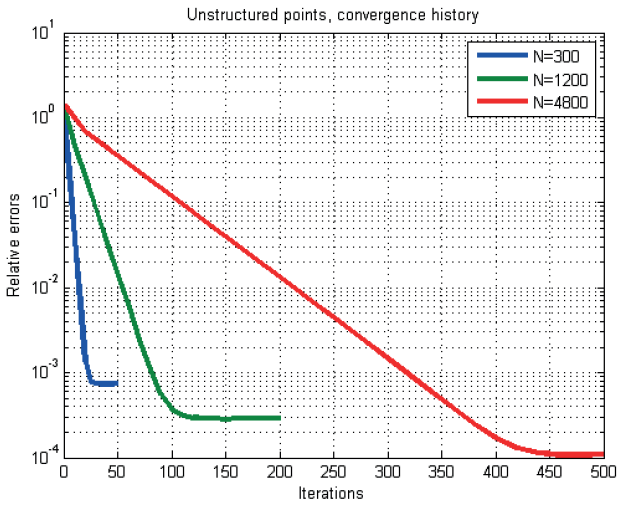


Fig. 5 Convergence history of the model problem for several numbers of local points. The rate of convergence decreases when the number of local points increases

order of magnitude, between cca. 12 and 40. It can clearly be seen that the rate of convergence decreases when N increases. This phenomenon is not surprising: the traditional Seidel-type iterations have the same property in structured grids. This indicates that the above iterative method should be combined with another technique in order to create a computationally efficient method.

Remark: The same technique can be applied also to the case when the local points are structured, e.g., they form a familiar rectangular computational grid. In the classical finite difference method, the proper handling of the boundary points is not trivial, since the computational grid does not fit a complicated boundary in general. A well-known technique is the Shortley-Weller approximation [12], which uses different schemes at the near-boundary points. This procedure may often be inconvenient; but using the proposed method, no such an additional problem arises. However, the overall rate of convergence may be low also in these structured cases.

The rate of convergence can be significantly increased by embedding the method into a multi-level context. Without going into deep details, we note that an analysis similar to the investigation of the convergence shows that the Schwarz iteration damps the high-frequency components of the error much more significantly than the low-frequency ones. Consequently, though the overall convergence rate is low, the method can serve as an efficient *smoothing procedure* of a multi-level technique. This will be outlined in the next section.

4 A quadtree-based multi-level technique

As is well known, every multi-level method requires a sequence of nested 'grids' (referred to as 'coarse levels' and 'fine levels'). At each level, the original problem is to be discretized. Inter-grid transfer operators are also required to make it possible the data transfer between the different levels. A smoothing procedure is also needed to damp the high-frequency error components. For the description of the classical multigrid methods, see e.g., [13].

The simplest multi-level technique is the *cascade method*. This is based on the following steps:

- On the coarsest level, solve the discrete problem exactly;
- Transfer the approximate solution to the next fine level;
- Apply several smoothing iterations on the actual level (the number of iterations can be kept under a moderate bound, independently of the actual level);
- Repeat the previous two steps until the finest level is reached.

A common advantage of the multi-level methods that the required computational cost (the number of the necessary arithmetic operations) remains proportional to the *first* power of the introduced unknowns only. This is much better than that of the classical solution techniques.

In the proposed method, the levels are defined by the quadtree (QT) algorithm controlled by the boundary of the original domain. As it is well known, this procedure results in a cell system, which is fine in the vicinity of the boundary and remains coarse in the middle. Fig. 6 shows a quadtree cell system generated by a circle. The local points in the applied multi-level method are defined to be the centers of the inner cells (completed by boundary collocation points). More precisely, the local points belonging to the kth level are defined to be the union of the quadtree cell centers belonging to the quadtree subdivision levels starting from a minimal level $k_{\min}$ (set to 4 in the later examples) until the actual level k. Thus, a monotonic increasing sequence of local point sets is obtained, the spatial distributions of which become dense toward the boundary and remain coarse in the middle of the domain. This property fits well the smoothness properties of harmonic functions, which are much smoother in the middle of the domain than in the vicinity of the boundary. Consequently, it is sufficient to use the above non-uniformly distributed local points defined by the QT algorithm.

4.1 Numerical examples

In all of the following examples, the model problem is the Laplace equation supplied with Dirichlet boundary condition consistent with the test solution:

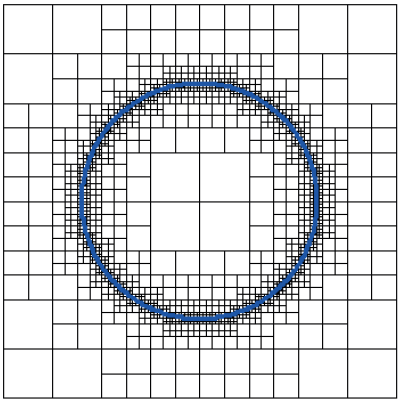


Fig. 6 A quadtree cell system generated by the points of a circle

$$u(x,y)=(x-0.5)^2-(y-0.5)^2\,. \tag{30}$$

The approximate solution method was the cascade method in all cases. Since the spatial distribution of the local points was highly nonuniform, the 'radius of influence' was defined to be level-dependent:

$$R:=\frac{4}{2^L}\,, \tag{31}$$

where L is the subdivision level of the cell, the center of which is the actual central point.

Example 2: Here the domain Ω is a circle centered at the point (0.5, 0.5) with radius 0.3. The total number of the local points is 676. The corresponding coarse and fine level points are shown in Fig. 7. Table 1 shows the relative L_2-errors at the different levels after performing only 15 iteration steps at each level, independently of the actual number of points. The example demonstrates that the computational complexity is significantly reduced due to the applied multi-level technique.

Example 3: Here Ω is a smooth star-shaped domain parametrized by the following pair of functions (see Fig. 8):

$$x=0.5+0.3\cdot(1+0.2\cdot\sin 5t)\cos t\,, \tag{32}$$

$$y=0.5+0.3\cdot(1+0.2\cdot\sin 5t)\sin t\,. \tag{33}$$

The total number of the local points is 794. The corresponding coarse and fine level points are shown in Fig. 9. Table 2 shows the relative L_2-errors at the different levels after performing 15 iteration steps at each level.

Example 4: Here Ω is an amoeba-shaped domain parametrized by the following pair of functions (see Fig. 10):

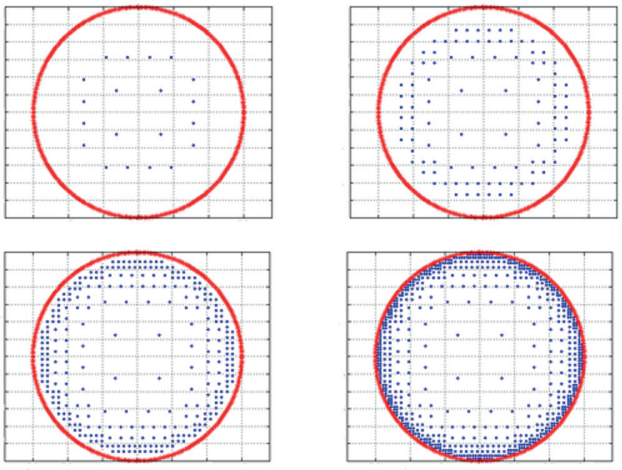


Fig. 7 Circle, the local points belonging to the subdivision levels 4, 5, 6 and 7

Table 1 Domain: circle. Relative L_2-errors of the approximate solution at the different levels

Level	4	5	6	7
Rel. L_2-errors	0.003294	0.0002276	0.0001176	0.00008035

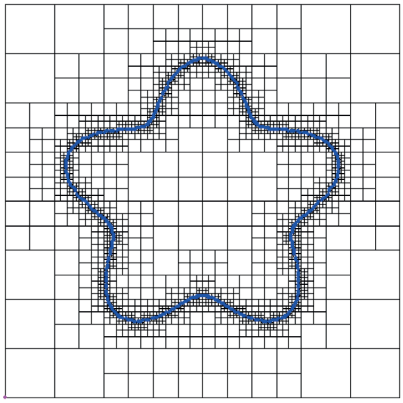


Fig. 8 Smooth star-shaped domain and quadtree cell system generated by the points of the boundary

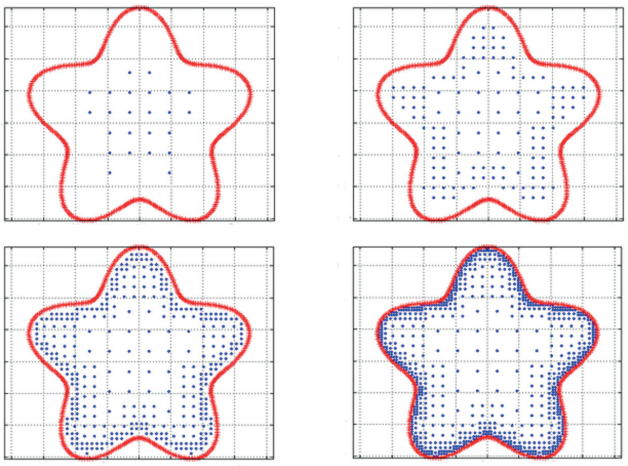


Fig. 9 Smooth star-shaped domain, the local points belonging to the subdivision levels 4, 5, 6 and 7

Table 2 Smooth star-shaped domain. Relative L_2-errors of the approximate solution at the different levels

Level	4	5	6	7
Rel. L_2-errors	0.004872	0.0008633	0.0004057	0.0002364

$$x=\frac{1}{6}(2.5+(e^{\sin t}\sin^2 2t+e^{\cos t}\cos^2 2t)\cos t)\,, \tag{34}$$

$$y=\frac{1}{5}(2.0+(e^{\sin t}\sin^2 2t+e^{\cos t}\cos^2 2t)\sin t)\,. \tag{35}$$

The total number of the local points is 758. The corresponding coarse and fine level points are shown in Fig. 11. Table 3 shows the relative L_2-errors at the different levels after performing 15 iteration steps at each level.

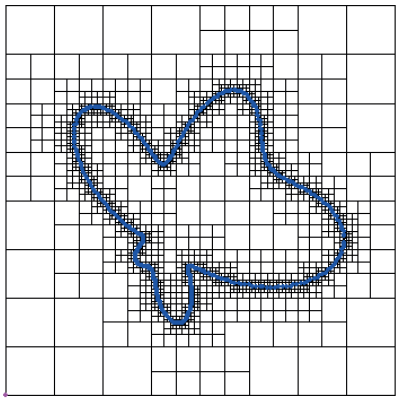


Fig. 10 Amoeba-shaped domain and quadtree cell system generated by the points of the boundary

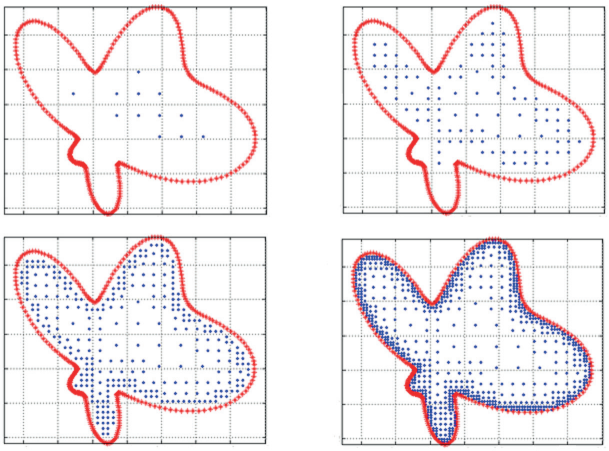


Fig. 11 Amoeba-shaped domain, the local points belonging to the subdivision levels 4, 5, 6 and 7

Table 3 Amoeba-shaped domain. Relative L_2-errors of the approximate solution at the different levels

Level	4	5	6	7
Rel. L_2-errors	0.004817	0.0005986	0.0002861	0.0001695

5 Conclusions

A localized version of the Method of Fundamental Solutions has been proposed. The localization splits the original problem into several subproblems defined on some subdomains of the original domain. These subproblems are solved by the Method of Fundamental Solutions using small numbers of local source points which are defined in the exterior of the subdomains. The subproblems are solved in an iterative way, which mimics the familiar Schwarz overlapping method. The rate of convergence has been speeded up by embedding the technique into a multi-level context. The coarse and fine levels of the multi-level method have been defined by a quadtree-generated cell system controlled by the boundary of the original domain. The method significantly reduces the necessary computational cost. In addition to it, the problem of solution of large linear systems with fully populated and severely ill-conditioned matrices is completely avoided.

References

[1] Golberg, M. A. "The Method of Fundamental Solutions for Poisson's Equation", Engineering Analysis with Boundary Elements, 16, pp. 205–213, 1995. https://doi.org/10.1016/0955-7997(95)00062-3

[2] Golberg, M. A., Chen, C. S. "The Method of Fundamental Solutions for Potential, Helmholtz and Diffusion Problems", In: Golberg, M. A. (ed.) Boundary Integral Methods: Numerical and Mathematical Aspects, WIT Press 1998, pp. 103–176. ISBN 1853125296

[3] Šarler, B. "Solution of potential flow problems by the modified method of fundamental solutions: formulations with the single layer and the double layer fundamental solutions", Engineering Analysis with Boundary Elements, 33, pp. 1374–1382, 2009. https://doi.org/10.1016/j.enganabound.2009.06.008

[4] Young, D. L., Chen, K. H., Lee, C. W. "Novel Meshless Method for Solving the Potential Problems with Arbitrary Domain", Journal of Computational Physics, 209, pp. 290–321, 2005. https://doi.org/10.1016/j.jcp.2005.03.007

[5] Gáspár, C. "A multi-level technique for the method of fundamental solutions without regularization and desingularization", Engineering Analysis with Boundary Elements, 103, pp. 145–159, 2019. https://doi.org/10.1016/j.enganabound.2019.03.006

[6] Gáspár, C. "Application of quadtrees in the method of fundamental solutions using multi-level tools", In: Alves, C., Karageorghis, A., Leitão, V., Valtchev, S. (eds.) Advances in Trefftz Methods and Their Applications, SEMA SIMAI Springer Series, 23, Springer, 2020, pp. 41–57. ISBN 978-3-030-52803-4

[7] Gáspár, C. "Global and Local Multi-level Meshless Schemes Based on Multi-Elliptic Interpolation", In: Leitão, V. M. A., Alves, C. J. S. Duarte, C. A. (eds.) Proceedings of ECCOMAS Thematic Conference on Meshless Methods, Lisbon, Portugal, 2005, pp. B12.1-B12.6. ISBN 972-99289-1-6

[8] Fan, C. M., Huang, Y. K., Chen, C. S., Kuo, S. R. "Localized method of fundamental solutions for solving two-dimensional Laplace and biharmonic equations", Engineering Analysis with Boundary Elements, 101, pp. 188–197, 2019. https://doi.org/10.1016/j.enganabound.2018.11.008

[9] Liu, S., Li, P.-W., Fan, C.-M., Gu, Y. "Localized method of fundamental solutions for two- and three-dimensional transient convection-diffusion-reaction equations", Engineering Analysis with Boundary Elements, 124, pp. 237–244, 2021. https://doi.org/10.1016/j.enganabound.2020.12.023

[10] Lions, P. L. "On the Schwarz Alternating Method I", In: Glowinski, R., Golub, J. H., Meurant, G. A., Periaux, J. (eds.) Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, Paris, France, SIAM, 1988, pp. 1–42. ISBN 0-89871-220-3

[11] Hu, J. W., Wan, C. H. "Rate of Convergence of Schwarz Alternating Method for Time-dependent Convection-diffusion Problem", Journal of Computational Mathematics, 20(5), pp. 479–490, 2002. [online] Available at: https://www.jstor.org/stable/43693013

[12] Matsunaga, N., Yamamoto, T. "Superconvergence of the Shortley-Weller Approximation for Dirichlet Problems", Journal of Computational and Applied Mathematics, 116, pp. 263–273, 2000. https://doi.org/10.1016/S0377-0427(99)00321-0

[13] Stüben, K., Trottenberg, U. "Multigrid methods: Fundamental algorithms, model problem analysis and applications", In: Hackbusch, W., Trottenberg, U. (eds.) Multigrid Methods. Lecture Notes in Mathematics, Vol. 960. Springer, 1982, pp. 1–176. https://doi.org/10.1007/BFb0069928