

Optimal Design of Planar Trusses Using Graph Theoretical Force Method

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Abstract

In this paper, the purpose is to present an efficient graph method for the analysis of truss structures using the force method and compare the computational time with that of the stiffness approach. Naturally, the results of the optimization and the accuracy of calculation for both methods are identical, but the calculation time for the force method is less when the degree of static indeterminacy (DSI) is smaller than the degree of kinematic indeterminacy (DKI) of the structure. Three examples are designed, and optimization has been performed using the ECBO algorithm in MATLAB.

Keywords

graph-theoretical force method, planar truss structures, metaheuristic algorithm, ECBO

1 Introduction

Analysis of structures can be performed by either force method or displacement approach. The force method is less developed due to its difficulty in generating a suitable statical basis, while the stiffness method is more amenable for computation. For this reason, some of the advantages of force method has been ignored in non-linear analysis and optimization, Kaveh [1].

Our criterion for selecting the structural analysis among these two methods is the degree of uncertainty of the structure. The DSI and DKI of a structure is calculated, and the one which is smaller the corresponding method is chosen for the analysis. (Force method for DSI and the displacement method for the case when DKI is smaller) [1].

In this paper, examples of trusses are designed that are analyzed by both force and displacement methods. The selected examples have a lower DSI than DKI, and thus the analysis time reduction and memory saving in the force method have been evident.

2 Force method

Force method or continuous deformation method is based on assuming the equilibrium to hold and proceed by satisfying the compatibility. In this method, internal forces are obtained considering some member forces and support reactions as redundant.

Five different approaches exist for the force method of structural analysis, which are known as:

1. Topological force methods,
2. Algebraic force methods,
3. Mixed algebraic-combinatorial force methods,
4. Integrated force method.
5. Graph-theoretical force methods.

Topological force methods have been developed by Henderson [2] and Henderson and Maunder [3] for rigid-jointed skeletal structures using manual selection of the cycle bases of their graph models [4]. Methods suitable for computer methods are due to Kaveh [5–7]. The topological methods are generalized to cover all types of skeletal structures, such as rigid-jointed frames, pin-jointed planar trusses and ball-jointed space trusses by Kaveh [8], Cassell, [9] and Kaveh [10].

Algebraic methods have been developed by Denke [11], Robinson and Haggemacher [12], Topçu [13], Kaneko et al. [14], Soyer and Topçu [15] and mixed algebraic-topological methods have been suggested by Gilbert and Heath [16], Coleman and Pothén [17, 18], and Pothén [19]. The integrated force method is due to Patnaik [20, 21], in which the equilibrium equations and the compatibility conditions are satisfied simultaneously in terms of the

force variables. Simultaneous analysis and design by force method can be found the work of Kaveh and Rahami [22].

2.1 Formulation

Consider a structure S with M members and N nodes, which is $\gamma(S)$ times statically indeterminate. Select $\gamma(S)$ independent unknown forces as *redundant*. These unknown forces can be selected from external reactions and/or internal forces of the structure [23]. Denote these redundant by:

$$q = \{q_1, q_2, \dots, q_{\gamma(S)}\}^t \quad (1)$$

Remove the constraints corresponding to redundant, in order to obtain a statically determinate structure, known as the *basic (released or primary) structure* of S . Obviously, a basic structure should be rigid. Consider the joint loads as,

$$p = \{p_1, p_2, \dots, p_n\}^t \quad (2)$$

where n is the number of components for applied nodal loads. Now the stress resultant distribution r due to the given load p for a linear analysis by the force method can be written as

$$r = B_0 p + B_1 q \quad (3)$$

Where B_0 and B_1 are rectangular matrices, each having m rows, and n and $\gamma(S)$ columns, respectively, m being the number of independent components for member forces. $B_0 p$ is known as a *particular solution*, which satisfies equilibrium with the imposed load and $B_1 q$ is a *complementary solution* formed from a maximal set of independent self-equilibrating stress systems (S.E.Ss), known as a *statical basis* [24].

Special and complementary solutions can be obtained from a basic structure, although it is not necessary to do so, and the special solution can be obtained from one basic structure and the complementary solution from another structure.

In fact, when the basic structure used in the special and complementary solutions are the same, it is equivalent to choosing the cycles of the structure from among the elementary cycles, although such a basis satisfies the equations and it is easy to find it, but it is not an effective method for analysis. Because we are looking for cycles that satisfy certain conditions [1].

Compatibility conditions are written as follows for each member using the deformation load relationship:

$$u = F_m r = F_m B_0 p + F_m B_1 q \quad (4)$$

where F_m is the stiffness of the unassembled matrix, which is obtained by superimposing the shape change load relations of each member in the diameter of the matrix.

The previous relation (Eq. (4)) can be written in the matrix form as

$$[u] = [F_m] [B_0 \ B_1] \begin{bmatrix} p \\ q \end{bmatrix} \quad (5)$$

Using the principle of virtual work, we have:

$$[V] = \begin{bmatrix} B_0^t \\ B_1^t \end{bmatrix} [u] \quad (6)$$

By previous combining relations (Eqs. (5) and (6)), one obtains the following relation:

$$\begin{bmatrix} V_0 \\ V_c \end{bmatrix} = \begin{bmatrix} B_0^t \\ B_1^t \end{bmatrix} [F_m] [B_0 \ B_1] \begin{bmatrix} p \\ q \end{bmatrix} \quad (7)$$

In relation (Eq. (7)), (V_0) includes the displacements caused by the applied loads, p , and V_c includes the change of the displacements of the cuts 1 in the base structure. By performing the operation, we have:

$$\begin{bmatrix} V_0 \\ V_c \end{bmatrix} = \begin{bmatrix} B_0^t F_m B_0 & B_0^t F_m B_1 \\ B_1^t F_m B_0 & B_1^t F_m B_1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \quad (8)$$

Thus

$$V_0 = (B_0^t F_m B_0) p + (B_0^t F_m B_1) q \quad (9)$$

$$V_c = (B_1^t F_m B_0) p + (B_1^t F_m B_1) q \quad (10)$$

The compatibility condition will indicate that the change of the displacements of the cuts should be equal to zero.

$$V_c = 0 \quad (11)$$

By combining two relations (Eqs. (10) and (11)), leads to

$$q = -(B_1^t F_m B_1)^{-1} (B_1^t F_m B_0) p \quad (12)$$

By substituting the obtained value for q in relation (Eq. (9)), we have:

$$V_0 = [B_0^t F_m B_0 - (B_1^t F_m B_1)^{-1} B_1^t F_m B_0] p \quad (13)$$

Now we get the stress distributions in the structure as

$$r = [B_0 - B_1 (B_1^t F_m B_1)^{-1} B_1^t F_m B_0] p \quad (14)$$

The structural flexibility matrix is often defined as $G = B_1^t F_m B_1$ [23-25].

2.2 Analysis process

1. Form B_0 and B_1 matrices.
2. Calculate the matrices $B_0^t F_m B_0$, $B_1^t F_m B_0$, $G = B_1^t F_m B_1$.
3. Calculate the matrix $G^{-1} = (B_1^t F_m B_1)^{-1}$.
4. Form the matrix $Q = (B_1^t F_m B_1)^{-1} B_1^t F_m B_0$, which is obtained from the inverse product of the matrix G in the matrix $B_1^t F_m B_0$.
5. By forming $B_1 Q$ and adding B_0 to it, the matrix $B = B_0 + B_1 Q$ is obtained.
6. Now the internal forces are obtained as $r = B_p$.

2.3 Optimal analysis of structures

For an optimal analysis by the force method, the flexibility matrix G should have the following properties [5]:

1. Sparse,
2. well-conditioned,
3. It should be well-structured (for example, it has a small bandwidth).

2.4 Force method for the analysis of planer trusses

In the force method for a truss, B_0 and B_1 should first be obtained. There are two graph theory methods to obtain these matrices. The first method uses associate graph, and the second method employs a bipartite graph [8]. In this article, the associate graph method is used. A similar method for the analysis has recently been applied to rigid-jointed frames by Kaveh and Zaerreza [26]

2.4.1 Associate graphs for selection of a suboptimal general cycle basis

The associate graph of S , denoted by $A(S)$, is a graph whose nodes are in a one-to-one correspondence with triangular panels of S , and two nodes of $A(S)$ are connected by a member, if the corresponding panels have a common member in S [8] and [27].

If S has some cut-outs, as shown in Example 3, then its associate graph can still be formed, provided that each cut-out is surrounded by triangulated panels.

For trusses containing adjacent cut-outs, a cut-out with cut-nodes in its boundary, or any other form violating the above-mentioned condition, extra members can be added to S . The effect of such members should then be included in the process of generating its self-equilibrating stress systems.

A maximal set of independent γ -cycles of S is defined as a *generalized cycle basis* (GCB) of S . A maximal set of independent fundamental γ -cycles, is termed a *fundamental generalized cycle basis* of S , Refs. [8] and [27]. For example, consider the truss shown in Fig. 1.

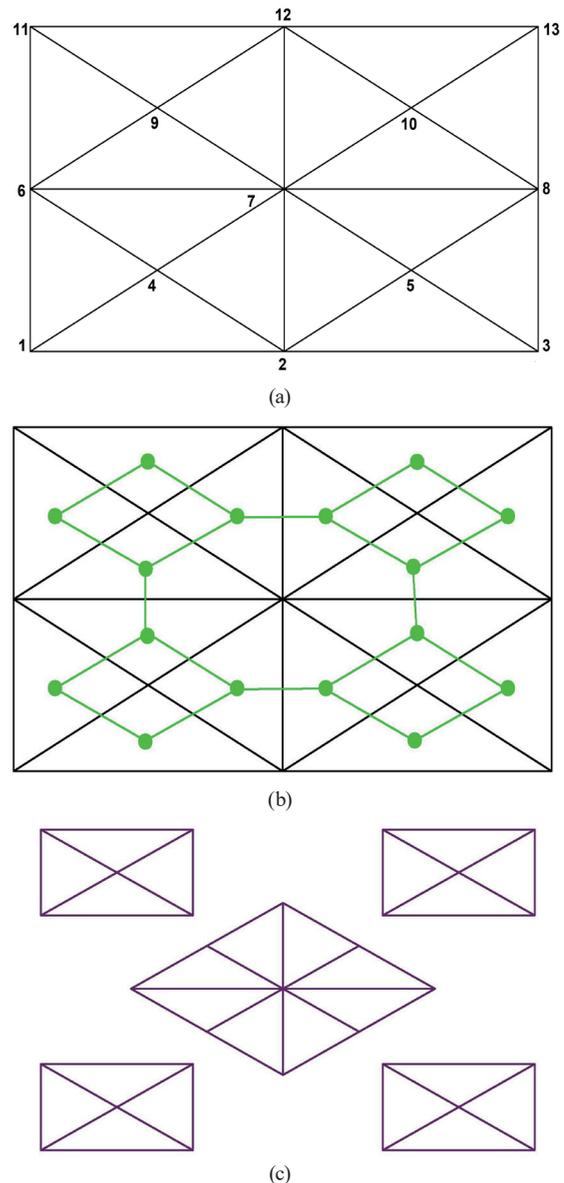


Fig. 1 A planar truss S , associated graph of S and the elements of a GCB of S ; a) A planar truss S , b) Associated graph of S , c) The elements of a GCB of S

The process of obtaining matrices B_0 and B_1 utilizing the associate graph method is as follows:

- Step 1: Construct the associate graph $A(S)$ of S .
- Step 2: Select a mesh basis of $A(S)$, using an appropriate cycle selection algorithm.
- Step 3: Select the γ -cycles of S corresponding to the cycles of $A(S)$.
- Step 4: Formation of self-equilibrium systems by extracted according to the cycles formed in the associated graph.
- Step 5: After the formation of self-equilibrium systems, a statical basis with localized self-equilibrating stress systems will be obtained [27].

3 Optimization algorithms

Different algorithms are available for optimization with various applications [26, 28–32].

3.1 ECBO

Colliding Bodies Optimization (CBO) was developed by Kaveh and Mahdavi [33] and improved by Kaveh and Ilchi Ghazaan [34] as the Enhanced Colliding Bodies Optimization (ECBO) which uses a memory to save a number of historically best CBs and utilizes a mechanism to escape from local optima.

The initial position of all objects (CBs) in an n-dimensional space is randomly determined:

$$x_k^0 = x_{MIN} + rand \times (x_{MAX} - x_{MIN}), \quad k = 1, 2, \dots, n. \quad (15)$$

Every collision object has a certain momentum, which is defined as follows:

$$Mass_k = \frac{1}{\sum_{k=1}^n \frac{1}{func(k)}}, \quad k = 1, 2, \dots, n. \quad (16)$$

To select the objects, CBs are sorted according to their mass in descending order, and we divide them into 2 equal groups:

1. Stationary objects,
2. Moving objects.

Moving objects collide with stationary objects to improve their position and move them to a better position. The speed of all objects before moving and collision is $V(i)$.

$$V_k = 0, \quad k = 1, 2, \dots, \frac{n}{2} \quad (17)$$

$$V_k = x_{k-\frac{n}{2}} - x_k, \quad k = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n \quad (18)$$

Velocity of static and moving objects after collision:

$$V'_k = \frac{\left(Mass_{k+\frac{n}{2}} + \varepsilon Mass_{k+\frac{n}{2}} \right) V_{k+\frac{n}{2}}}{Mass_k + Mass_{k+\frac{n}{2}}}, \quad k = 1, 2, \dots, \frac{n}{2} \quad (19)$$

$$V'_k = \frac{\left(Mass_k - \varepsilon Mass_{k-\frac{n}{2}} \right) V_k}{Mass_k + Mass_{k-\frac{n}{2}}}, \quad k = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n, \quad (20)$$

$$\varepsilon = 1 - \frac{iter}{iterMax}. \quad (21)$$

New position of objects:

$$x_k^{new} = x_k + random \times V'_k, \quad k = 1, 2, \dots, \frac{n}{2}, \quad (22)$$

$$x_k^{new} = x_{k-\frac{n}{2}} + random \times V'_k, \quad k = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n. \quad (23)$$

The parameter *pro* is selected from the interval (0, 1). This parameter determines whether a component of each CB should be changed or not. For each collision object, *Pro* is compared with rn_i . Here, rn_i is a random number between (0,1) which is uniformly distributed. If $rn_i < pro$, a dimension of the CB is randomly selected, and its value is reconstructed as follows:

$$x_{ij} = x_{j,min} + rand \times (x_{j,MAX} - x_{j,MIN}). \quad (24)$$

Flowchart of the ECBO is shown in Fig. 2, [33].

In the current optimization problem, the goal is to minimize the weight of the steel used while satisfying the regulatory limits of member tension and member slenderness. In addition to stress, there are several behavioral constraints that must be evaluated according to the design code. This is a numerically cumbersome operation, especially for large structures when no analytical method can be applied directly. On the other hand, the existing structural profiles are practically limited to a specific discrete list, which

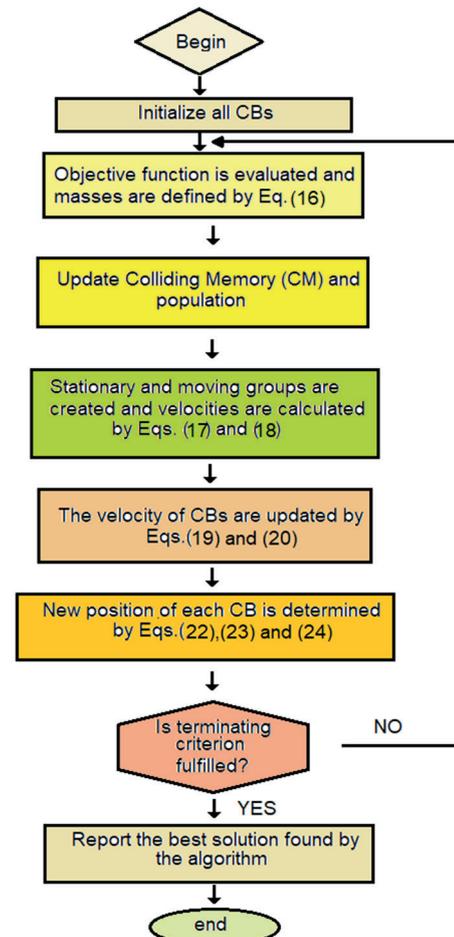


Fig. 2 Flowchart of the ECBO algorithm [33]

generally makes the discrete optimization problem more complicated than the continuous one. For this purpose, the problem has been specialized with discrete variables, i.e., the cross-section number of the truss members, and programmed in the MATLAB environment. The continuous values that may appear during the execution are rounded to correct variables and decoded before analyzing the structure. The vector of design variables for a structure with member group is given according to relation (Eq. (25)):

$$X = \langle x_i \rangle, \quad i = 1, 2, \dots, m. \quad (25)$$

Then, the volume of the structure is calculated from the internal multiplication of the length vector of the members, that is, L , in their cross-sectional area A , and the optimization of the weight of the skeleton is formulated with the help of Eq. (26) with the density under constraints.

$$\text{Min } w(X) = \rho L' \times A(x) \quad (26)$$

$$\text{S.t. } \begin{cases} g_j(x) \leq 0 \\ x_{imin} \leq x_i \leq x_{imax} \end{cases} \quad (27)$$

The functions including the limits of stress, displacement and allowable slenderness are presented in the form of relations (Eqs. (28) to (29)).

$$g_{\sigma}^k = \frac{\sigma_k}{(\sigma_k)_{allowable}} - 1 \leq 0, \quad k = 1, 2, \dots, N_d \quad (28)$$

$$g_{\lambda}^k = \frac{\lambda_k}{(\lambda_k)_{allowable}} - 1 \leq 0, \quad k = 1, 2, \dots, N_d \quad (29)$$

σ is member tension, N_d is number of members and λ is member slenderness.

4 Examples

In this section, 3 examples of planar trusses are designed and analyzed by the force and displacement methods. The examples have a lower DSI than DKI, and as shown in the

Table 1 List of the available sections for the trusses models

Section ID	Section Name	Area (10^{-4}m^2)	Section ID	Section Name	Area (10^{-4}m^2)
1	W12x26	49.35	19	W10x77	145.81
2	W8x28	53.23	20	W12x79	149.68
3	W12x30	56.71	21	W12x87	165.16
4	W8x31	58.9	22	W12x96	181.94
5	W10x33	62.65	23	W14x99	187.74
6	W8x35	66.45	24	W10x100	189.68
7	W10x39	74.19	25	W21x101	192.26
8	W8x40	75.48	26	W27x102	193.55
9	W14x43	81.29	27	W24x103	195.48
10	W12x45	85.16	28	W24x104	197.42
11	W8x48	90.97	29	W12x106	201.29
12	W10x49	92.9	30	W12x120	227.74
13	W12x53	100.64	31	W12x136	257.41
14	W10x54	101.94	32	W12x152	288.38
15	W12x58	109.68	33	W12x170	322.58
16	W10x60	113.55	34	W12x190	360
17	W12x65	123.23	35	W12x210	398.69
18	W12x72	136.13			

following results, the analysis time reduction and memory saving in the force method are evident.

4.1 Example 1: A 96-bar planar truss

The 96-bar plane truss is shown in Fig. 3. The number of nodes of the truss is 43. The elastic modulus is 210 GPa. The bars are categorized into 11 groups. For this truss the $DSI = m - 2n + 3 = 13$ and $DKI = 2n - 3 = 83$. The number of cycles in the associated graph for this truss is 13, and the dimensions of B_1 matrix are 96×13 . The number of degrees of freedom for the applied loading is 13. Thus, the dimensions of the B_0 matrix are also 96×13 . Table 1 shows the list of the available sections for the trusses. Fig. 4 shows the truss with its associate graph.

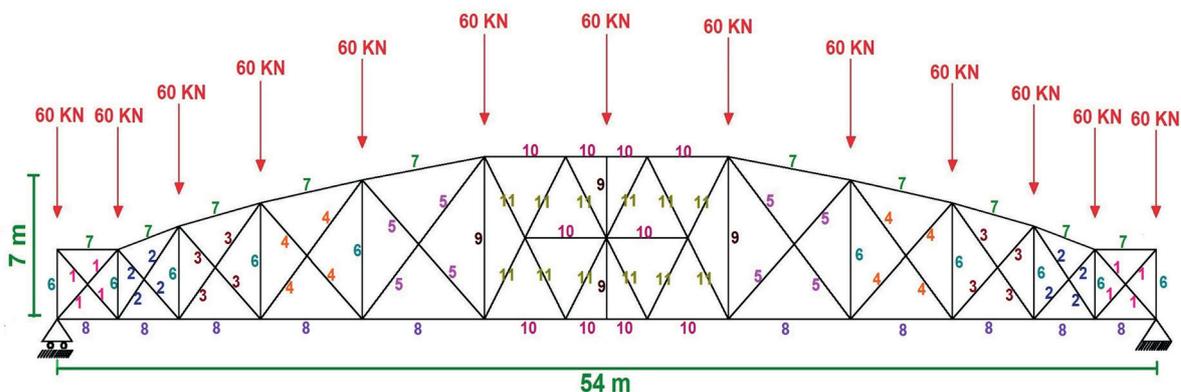


Fig. 3 Geometry and the member grouping of the 96-bar truss

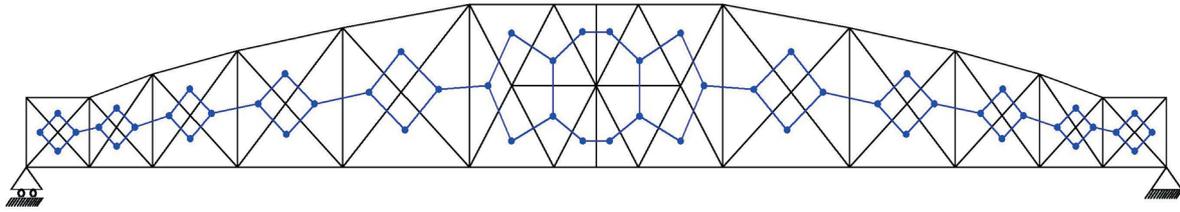


Fig. 4 A 96-bar truss and the associated graph of the truss

Fig. 5 shows the pattern of $B_1 \times B_1'$ matrix, the dimensions of this matrix are 96×96 , and its non-zero number is 968. The non-zero number of B_1 matrix is 114.

Optimization with ECBO algorithm was performed for the analysis of the truss with two methods of force and displacement, and as expected, similar answers were obtained. The convergence diagrams of the responses are shown in Fig. 6, and the optimum list of sections and results are shown in Table 2 for the 96-member truss.

Comparison of optimization time with two methods of forces and displacement are obtained as shown in Fig. 7. The optimization time with the force method is significantly less than that of the displacement method.



Fig. 5 Pattern of $B_1 \times B_1'$ matrix for the 96-bar truss with 968 non-zero entries

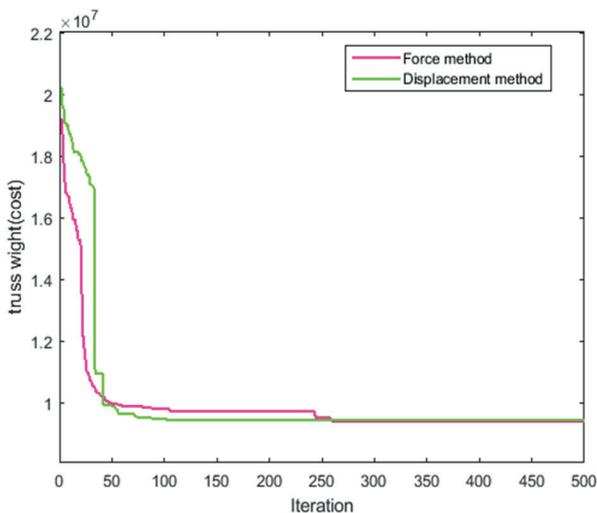


Fig. 6 Convergence curves for optimization of the 96-bar truss

Table 2 Results of the optimization for the 96-bar truss

Group number	Displacement method No. Section	Force method No. Section
of Element	23	34
1	15	17
2	18	12
3	13	11
4	1	1
5	18	20
6	34	35
7	28	31
8	25	11
9	35	35
10	1	1
11	9417810	9400850
Best weight (kg)	10225500	10361000
Average weight (kg)	807690	960150
Std (kg)		

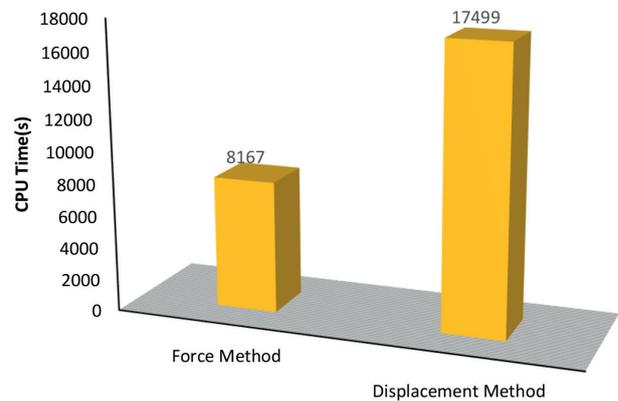


Fig. 7 Comparison of the time for optimization using force method and displacement method for the 96-bar truss

4.2 Example 2: A 194-bar planar truss

The 194-bar plane truss is shown in Fig. 8. The number of nodes of the truss is 81. The elastic modulus is 210 GPa. The bars are categorized into 8 groups. The $DSI = 35$ and $DKI = 159$. Considering that the number of regional cycles in the associated graph for this truss is 35, then the dimensions of B_1 matrix are 194×35 . The number of degrees of freedom in which the loading is applied is 12. Thus, the dimensions of the B_0 matrix are also 194×12 . Fig. 9 shows the truss with its associate graph.

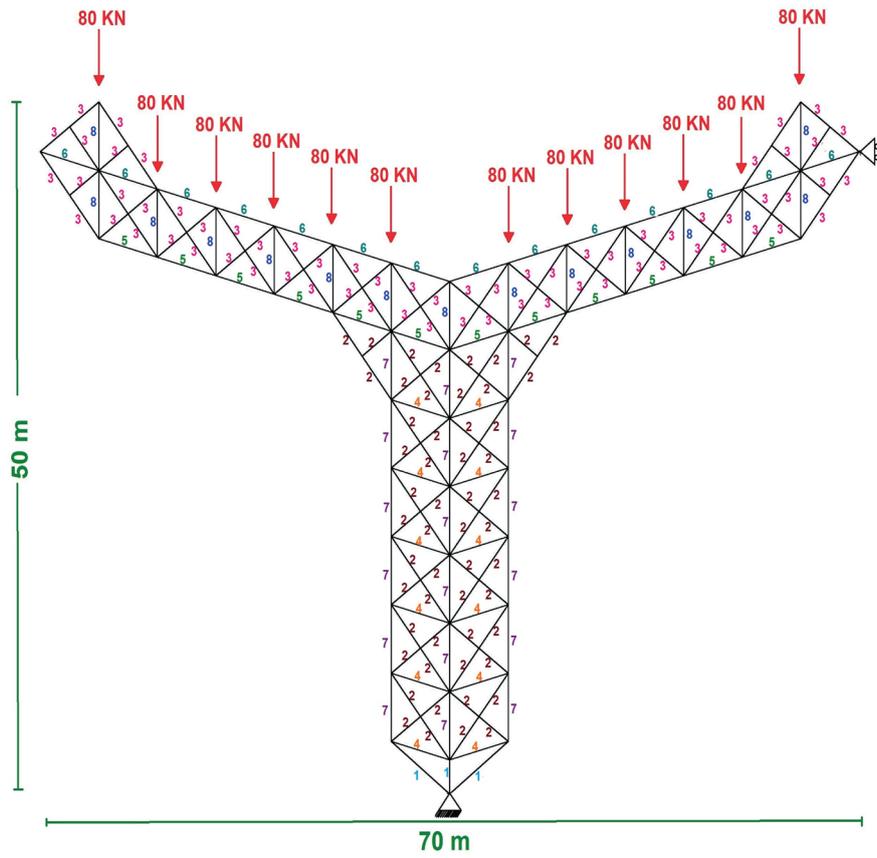


Fig. 8 Geometry and member grouping of the 194-bar truss

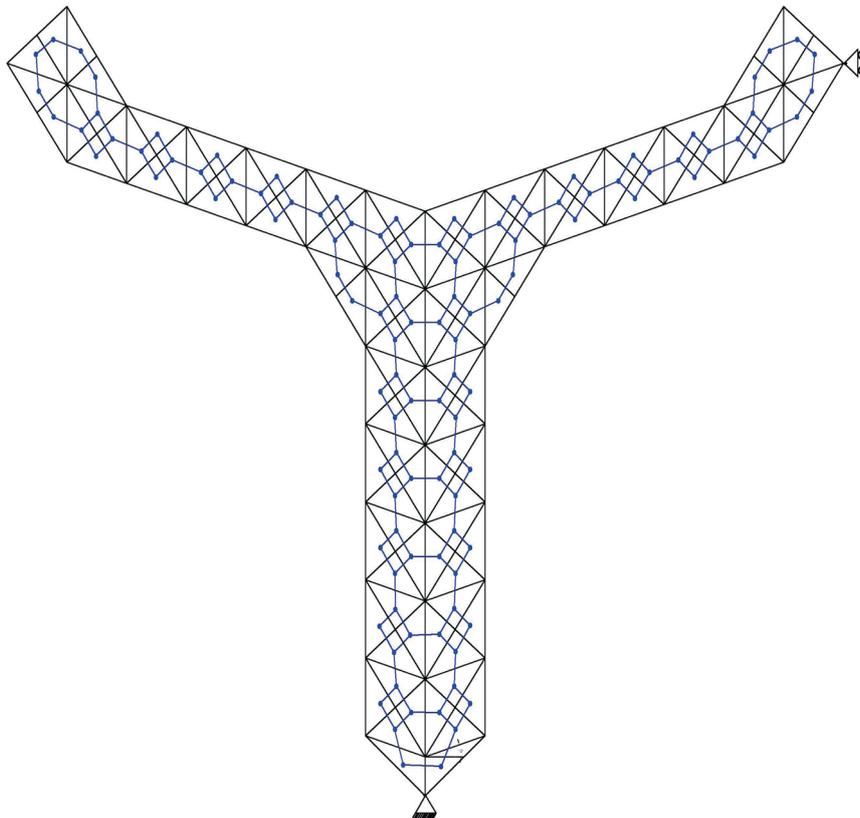


Fig. 9 A 194-bar truss and the associated graph of this truss

The self-equilibrium systems extracted by forming the associated graph and used to obtain matrix B_1 as shown in Fig. 10.

The pattern of $B_1 \times B_1'$ matrix, the dimensions of this matrix are 194×194 and its 2588 non-zero entries are shown in Fig. 11.

Results of the convergence for the ECBO algorithm with 500 iterations and 50 population number for the 194 members truss are shown in Fig. 12 with both analysis methods. The optimal sections and results are provided in Table 3.

In the 194-bar truss, as in the previous example, the time to perform the analysis with the force method was much less than that of the displacement method, which is shown graphically in Fig. 13.

4.3 Example 3: A 311-bar planar truss

The third example in Fig. 14 is a truss with 311 members and 131 nodes. Fig. 15 shows the truss with its associate graph. This example has a fundamental difference from the previous two examples, and that is the existence of a cut out

in the truss. The presence of a cut out that surrounded by triangular plates forms a type of cycle. Since of $A(S)$ corresponds to one γ -cycle of S , it is called a type I cycle, denoted by CI. Typical γ -cycles of S are shown by continuous lines,

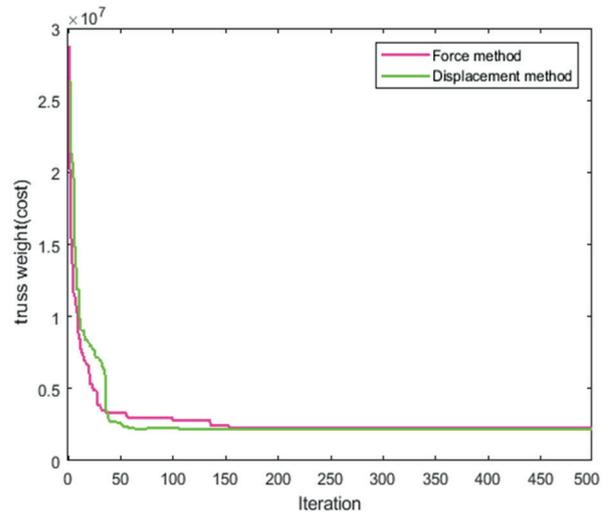


Fig. 12 Convergence curves for the 194-bar truss using ECBO

Table 3 Optimization results for 194-bar truss

Group number Of Element	Displacement method No. Section	Force method No. Section
1	35	35
2	35	35
3	18	19
4	35	35
5	28	28
6	10	12
7	33	32
8	14	14
Best weight (kg)	2195620	2280400
Average weight (kg)	2258400	2298120
Std (kg)	62780	17720

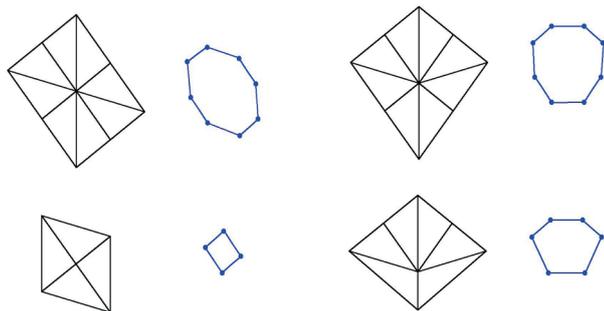


Fig. 10 Samples of self-equilibrium systems for the 194-bar truss

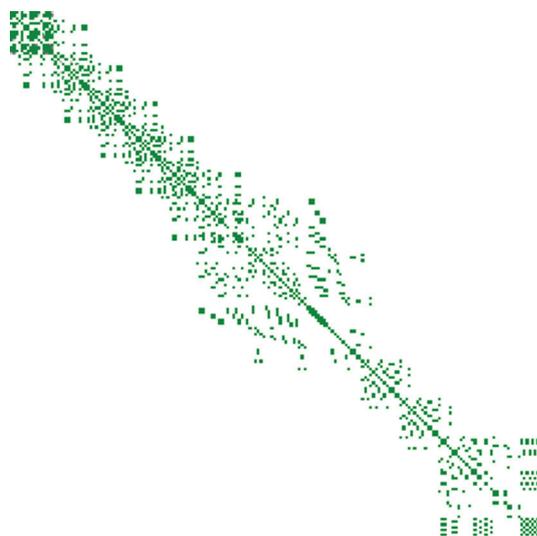


Fig. 11 Pattern of $B_1 \times B_1'$ matrix for the 194-bar truss with 2588 non-zero entries

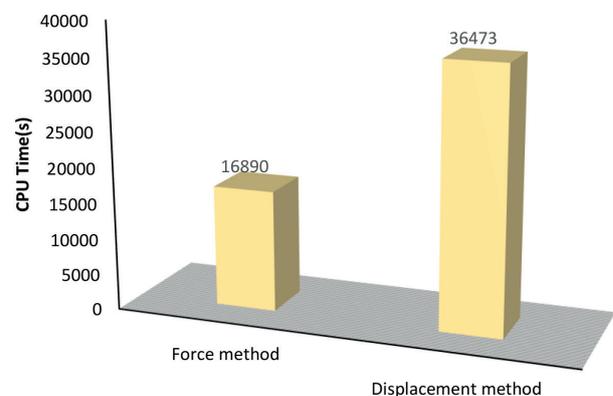


Fig. 13 Comparison diagram of optimization time in the force method and displacement method in the 194-bar truss

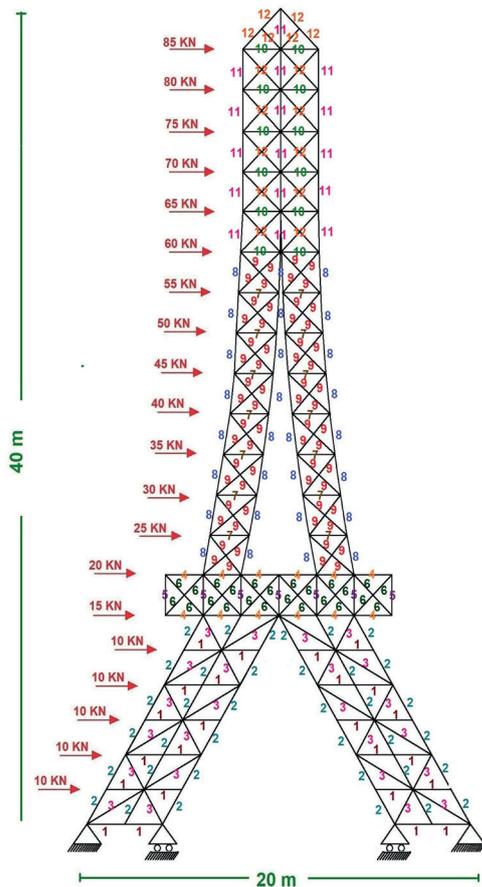


Fig. 14 Geometry and member grouping of the 311-bar truss

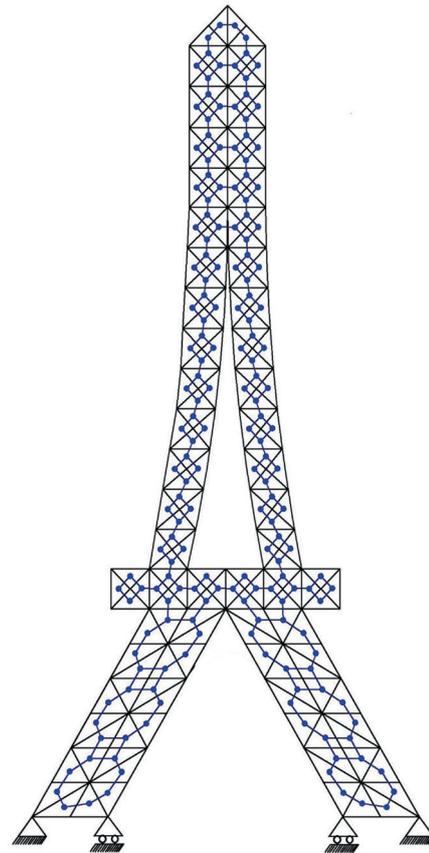


Fig. 15 A 311 bar truss and associated graph of the 311-bar truss

and their γ -chords are depicted in dashed lines, it is called a type III cycle, denoted by *CIII*. In Fig. 16(a), a triangulated truss and its associate graph, are shown and Fig. 16(b) shows a truss unit with one cut-out.

In this example, $DSI = 55$ and $DKI = 259$. Therefore, the force method is preferred in this example. Loads are applied at 20 nodes as shown in Fig. 16. Thus, the dimensions of the B_0 matrix are also 311×20 . The number of *CI* cycles in the associated graph for this truss is 49 and *CIII* cycles is 3, and we also have 3 degrees of external indeterminacy, then the dimensions of B_1 matrix are 311×55 . The elements of this truss, shown in the Fig. 16, are divided into 12 groups.

Self-equilibrium systems are extracted using to the cycles formed for the associated graph of the truss as shown in Fig. 17.

Fig. 18 shows the pattern of $B_1 \times B_1'$ matrix. The dimensions of this matrix are 311×311 and its non-zero entries is 17287.

The results of the convergence using ECBO algorithm for the 311-members truss is shown in Fig. 19 with both analysis methods. The optimal sections and the other results are given in Table 4.

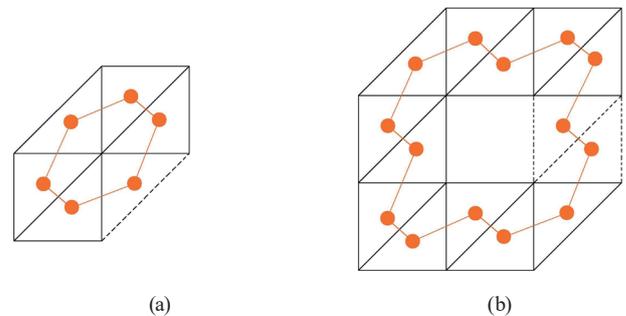


Fig. 16 Two different types of cycles; (a) A type *CI* cycle, (b) A type *CIII* cycle

CPU time for the truss with 311 members was also obtained for both analysis methods, which was significantly lower for the force method than the displacement method for this truss are similar to the previous two examples which are shown graphically in Fig. 20.

5 Conclusions

The purpose of this article is to compare two structural analysis methods for trusses, force method and displacement method. The force method in this article is performed using graph theory and utilizing the associate

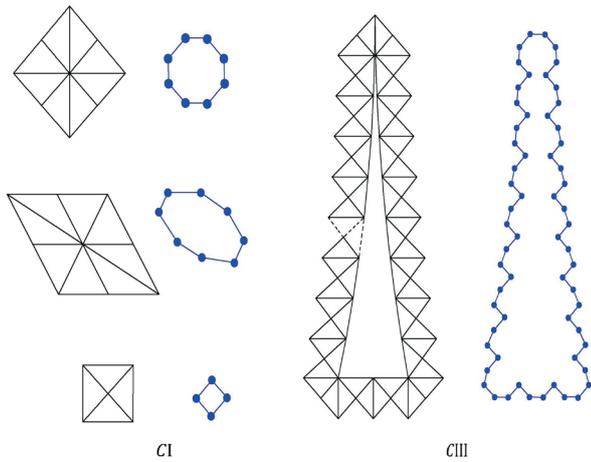


Fig. 17 Sample of self-equilibrium systems of the 311-bar truss

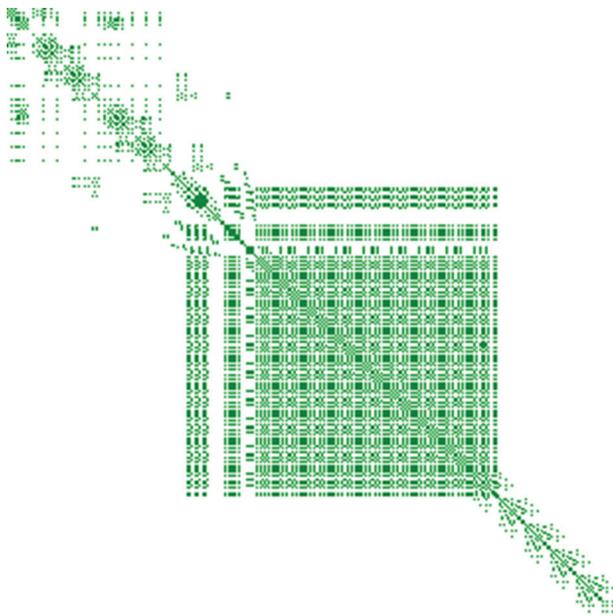


Fig. 18 Pattern of $B_1 \times B_1'$ matrix for the 311-bar truss with 17287 non-zero entries

graph. The structural weight was minimized subject to LRFD constraints of trusses. For this purpose, ECBO optimization algorithm was utilized to deal with section indices as discrete design variables. The optimal weight values for all 3 trusses were close to each other for both methods. However, the noteworthy point was the comparison of CPU time in two analysis methods. The amounts of CPU time in all 3 trusses with the force method were less than the displacement method, and with the increase in the difference between DSI and DKI, the difference in CPU time is increased. The comparison of these results is provided in Fig. 21.

Conflict of interest

No potential conflict of interest was reported by the authors.

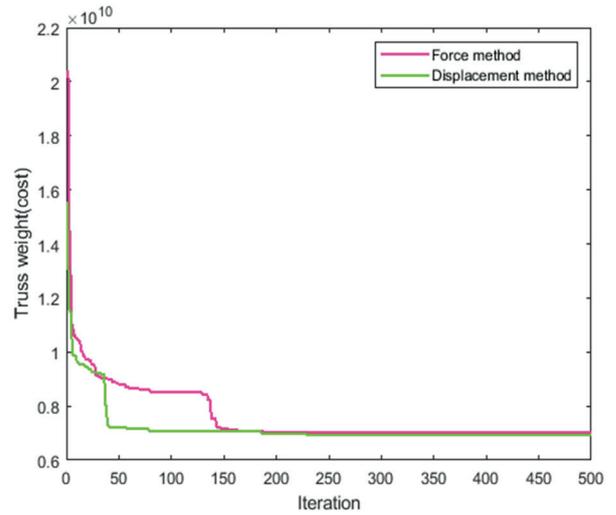


Fig. 19 Convergence curves for optimization of the 311-bar truss

Table 4 Results of the optimization for the 311-bar truss

Group number of Element	Displacement method No. Section	Force method No. Section
1	19	30
2	35	35
3	35	34
4	34	35
5	35	35
6	30	31
7	17	30
8	35	35
9	18	29
10	20	21
11	35	35
12	1	1
Best weight (kg)	6935518451	7011771617
Average weight (kg)	7024930000	7265420000
Std (kg)	89411549	253648383

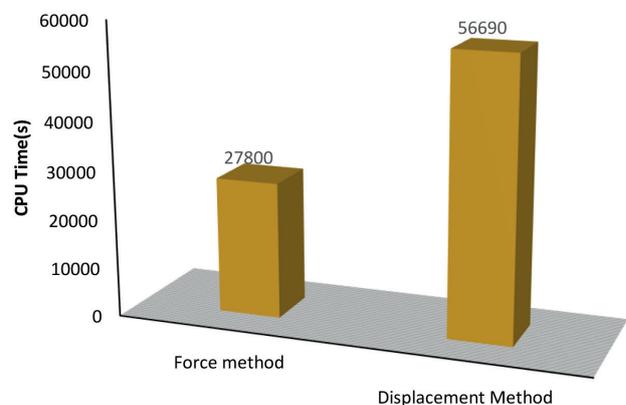


Fig. 20 Comparison of the optimization time in force method and displacement method for the 311-bar truss

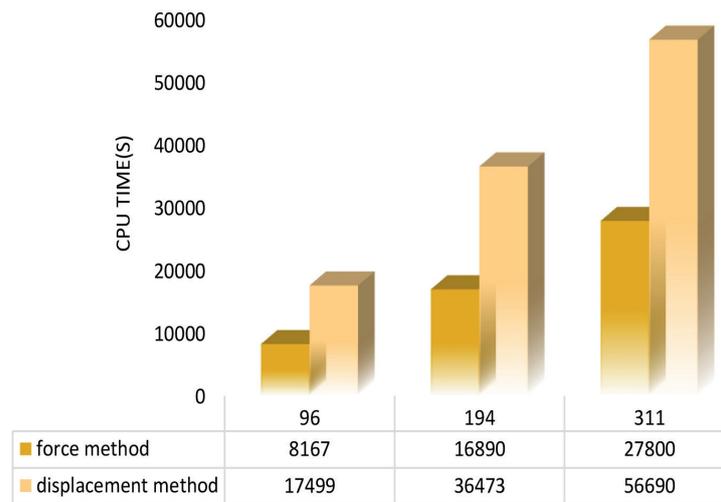


Fig. 21 Comparison of optimization time in the force method and the displacement method for three trusses

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